Anatomy of a Transmission Line Loudspeaker

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Introduction :

For the past several years, I have been performing transmission line loudspeaker simulations using MathCad[1] computer models. When I started deriving the transmission line equations of motion I wanted to be able to easily simulate tapered, straight, and expanding geometries. To accommodate the geometry variable in the derivation, I merged the one dimensional exponential horn wave equation with a fiber damping correlation I had empirically formulated. The fiber damping model was generated using data acquired from measurements of the electrical impedance and the SPL output of a straight test transmission line. The details of the transmission line modeling can be found on my website (www.quarter-wave.com).

There have been two or three major revisions to my MathCad transmission line models in the past couple of years. Some of the changes corrected errors in the derived equations while others extended the capabilities of the worksheets to represent additional types of enclosures. Since September of 2000, versions of the MathCad worksheets have been available for downloading and have received wide use by transmission line and TQWT DIY speaker building enthusiasts. To the best of my knowledge, the speakers built based on these MathCad worksheets have been very successful and performance measurements correlate extremely well with the MathCad simulation predictions.

While people have been applying my calculation methods to design transmission line loudspeakers for several years, there are always questions posted on Internet discussion forums or asked directly through private e-mails concerning how the transmission line enclosure actually functions. For example, questions concerning the electrical impedance curve and if it is a double or a single humped curve are often posed. A second significant point of discussion revolves around the resonances of the air column and the quarter wavelength frequencies and standing wave shapes. Over the years I have provided my responses to these two questions, and also to a host of other inquiries, by responding on discussion forums or by return e-mail. I decided recently that a useful document might be one that pulls some of these responses together and expands them to describe the inner workings of a transmission line enclosure. The result is this article which I have titled *Anatomy of a Transmission Line Loudspeaker*. I hope that the reader finds it both interesting and informative.

Displacement, Velocity, and Acceleration :

Three expressions need to be defined to allow the discussion to proceed and hopefully have some physical meaning for most readers. In the following paragraphs sinusoidal displacement, velocity, and acceleration will be referred to over and over again. For example, the equations describing one possible sinusoidal displacement function and its relationship to velocity and acceleration are shown below.

$$\text{displacement} = x(t) = \sin(\omega\, t)$$

$$\text{velocity} = u(t) = dx/dt = \omega \cos(\omega\, t) = \omega \sin(\omega\, t + 90 \text{ degrees})$$

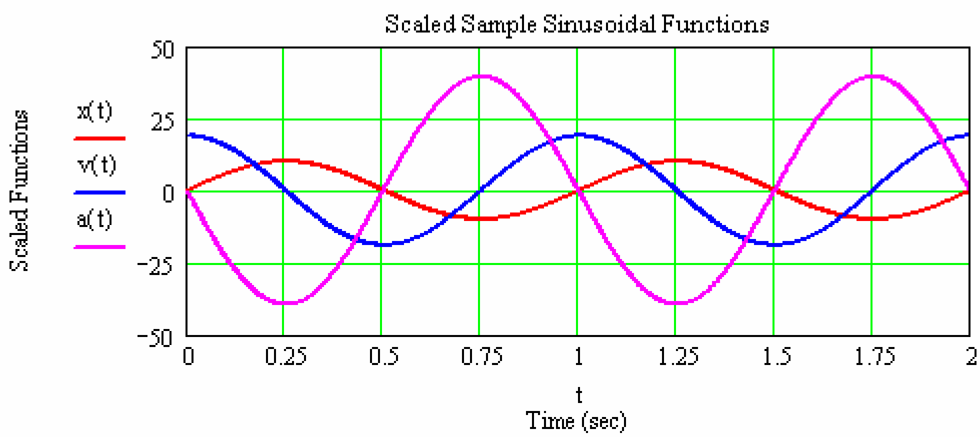
$$\text{acceleration} = a(t) = d^2x/dt^2 = -\omega^2 \sin(\omega\, t) = \omega^2 \sin(\omega\, t + 180 \text{ degrees})$$

In all of these expressions, the frequency ω has units of radians per second and is related to the more commonly used frequency variable f, having units of Hz or cycles per second, by the following expression.

$$\omega = 2\pi f$$

Reviewing the expressions for displacement, velocity, and acceleration; it is observed that each time you differentiate with respect to time the magnitude increases by a factor of ω and the phase shifts by 90 degrees. To help visualize a sinusoidal displacement, and the act of shifting it on the time axis by 90 degrees and 180 degrees to produce velocity and acceleration respectively, a sample plot is shown below in Figure 1. The magnitudes of the displacement and velocity curves have been scaled to produce a more easily visible comparison.

Figure 1 : Displacement, Velocity, and Acceleration Curves
Red curve = displacement = x(t)
Blue curve = velocity = u(t)
Magenta curve = acceleration = a(t)



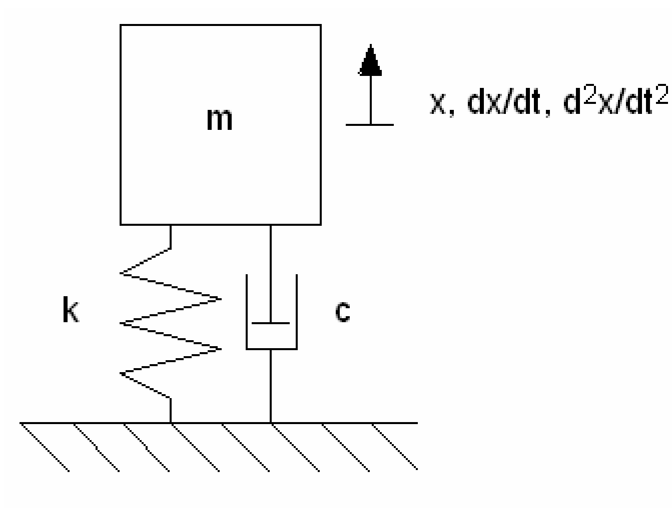
For the plot shown in Figure 1, the frequency was set equal to 1 Hz so that 90 degree and 180 degree phase shifts correspond to 0.25 and 0.5 seconds respectively on the time axis. Notice that the blue velocity curve is 90 degrees out of phase and that the magenta acceleration curve is 180 degrees out of phase with the red displacement curve. Recognizing the magnitude and phase relationships between the displacement, the velocity, and the acceleration is a key point to take away from this section. Frequently consulting the curves shown in Figure 1 should help when trying to understand what is being described in subsequent sections.

Mechanical Vibration Theory :

One Degree of Freedom System :

To appreciate how a transmission line enclosure works, a little background information about mechanical vibration theory is required. The starting point for this section is the single degree of freedom system as shown in Figure 2. A single degree of freedom system is comprised of a mass, a damper, and a spring. Motion is allowed only in a single direction, in this case the vertical direction. The mass moves up and down at one end of the spring and damper while the other ends are anchored to ground. The mass has a displacement, velocity, and acceleration as described in the preceding section. The positive sign convention is denoted on the right side of the figure by the arrow pointing upward.

Figure 2 : Single Degree of Freedom Mechanical System



The equation of motion for the single degree of freedom system is shown below. Notice that this equation sums acceleration, velocity, and displacement terms to react a forcing function on the right side of the equals sign. For a unit sinusoidal forcing function, F is equal to 1 Newton.

$$m\, d^2x/dt^2 + c\, dx/dt + k\, x = F \cos(\omega\, t)$$

In simple mechanical models typically a degree of freedom (or motion opportunity) is assigned to each lumped mass in the system. For the system shown in Figure 2, there is one mass hence a single degree of freedom and a single natural frequency. As the number of masses increases, so do the number of degrees of freedom and therefore the number of natural or resonant frequencies. For a continuous system, such as a column of air vibrating in the axial direction, the number of masses is infinite as are the number of resonant frequencies.

An undergraduate level vibration textbook[2, 3] will contain the complete derivation and solution of the differential equation of motion for the single degree of freedom system, so it will not be repeated. Instead, a sample problem will be used to calculate and plot the response of the mass when it is subjected to a 1 Newton sinusoidal excitation sweeping the frequency range from 10 Hz to 100 Hz. To set up and perform the numerical simulation, a few key variables must first be defined.

Assume the mass is 25 gm and that the system's natural frequency is 40 Hz. The natural frequency of a single degree of freedom system is calculated using the well-known equation shown below.

$$f_n = 1/2\pi \, (k / m)^{1/2}$$

Knowing the mass and frequency allows the spring rate to be calculated. The calculated spring rate is 1579 Newton/m. Finally, the damping value is set to produce a Q of 50 at the 40 Hz resonance. This is a very lightly damped mechanical system that can be subjected to a 1 Newton sinusoidal excitation and numerically solved at discrete frequencies using MathCad. This is the process I used to perform all of the simulations presented in this section.

Figures 3, 4, and 5 show the displacement, velocity, and acceleration responses for this single degree of freedom system subjected to a unit sinusoidal excitation. There are several classic properties of each response curve that will be described in the following paragraphs. But before diving into the details of each particular curve, it should be noted that this is a linear analysis. Basically, this means that if the excitation force were doubled the displacement, velocity, and acceleration response would also double. If the excitation force were halved, the response would also be halved. The numerical response, extracted from the curves shown in Figure 3, 4, or 5, can be scaled to determine the response of the system to any magnitude of sinusoidal forcing function.

One other feature common to all three response plots is the system behavior at resonance. A sharp peak at 40 Hz is seen in all three of the response plots. This is typical of a lightly damped mechanical system. At the resonant frequency, a small excitation level will produce a greatly amplified response when compared to other frequencies. What is sometimes neglected is the sudden phase shift that occurs as the forcing function sweeps through the resonant frequency. In each response plot, a 180 degree phase shift occurs as the frequency sweep traverses the resonant frequency of 40 Hz.

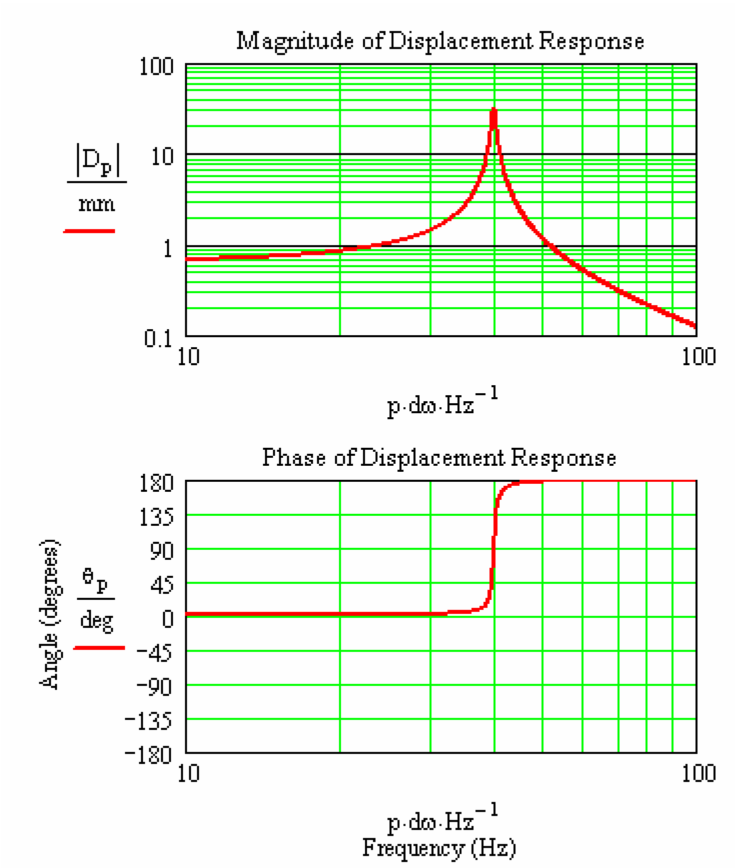
One example of a resonance that everybody has heard at one time or another is the tone produced by rubbing a crystal wine glass typically used at a formal reception. If you wet your finger and gently trace the perimeter of the glass, a ringing sound is generated that can be heard across the room. The glass resonates producing a tone that is a function of the size and shape of the glass and the amount of liquid remaining in the bottom of the glass. The response produced is grossly out of proportion to the excitation provided by your wet finger. If you rubbed your wet finger on one of the tables or chairs at the reception, with a similar pressure, chances are the sound produced would be inaudible even to you. The significant sound level produced by the wine glass is due to the excitation of a lightly damped resonance.

Figure 3 shows the displacement response for the unit sinusoidal excitation as it is swept from 10 Hz to 100 Hz. At low frequencies, the displacement is approaching a constant magnitude and the phase angle is zero. This is indicative of the forcing function working only against the spring without any influence from the mass or the damper. Calculating the deflection of the system's spring, due to a 1 Newton static force, produces the following value.

$$f / k = (1 \text{ Newton}) / (1579 \text{ Newton/m}) = 0.633 \text{ mm}$$

The magnitude of the displacement at 10 Hz is 0.674 mm and the trend is decreasing at lower frequencies. At 1 Hz, the displacement value is approximately 0.633 mm. This means that at low frequencies the response is stiffness controlled.

Figure 3 : Displacement Response per Newton of Excitation

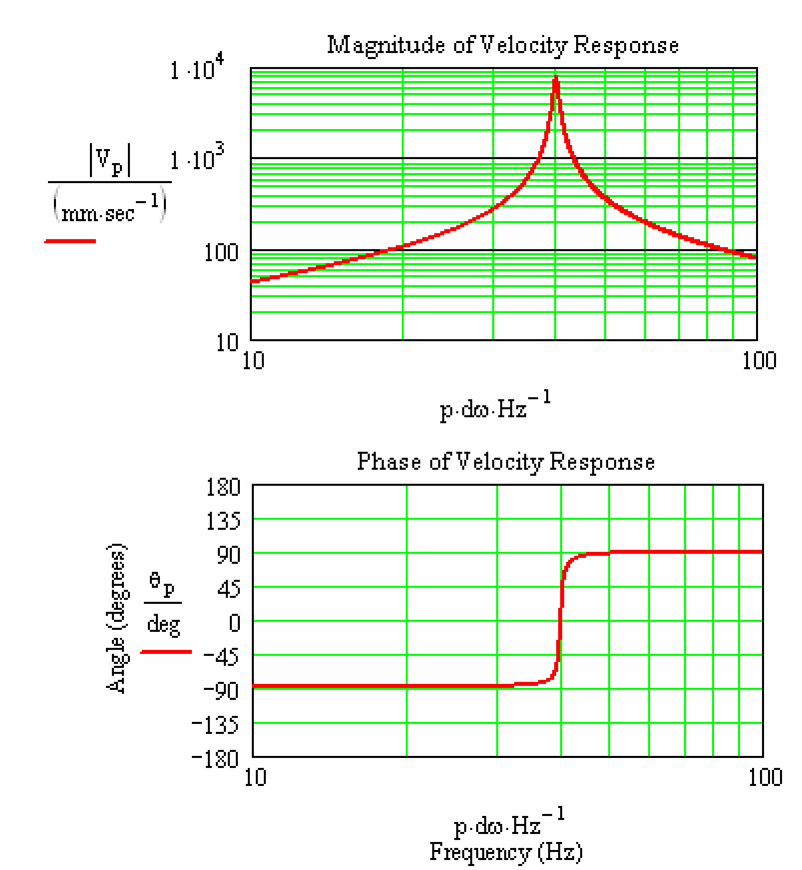


A second distinctive feature of the plot is the sharp peak at the resonant frequency of 40 Hz with a corresponding phase of 90 degrees. The velocity term in the equation of motion determines the magnitude of this peak. For this lightly damped system, the response peak magnitude is 31.663 mm, which as expected is approximately a factor of 50 (the system Q) greater than the low frequency response magnitude of 0.633 mm.

The final feature to be noted is the negative slope of the magnitude plot and the corresponding phase angle above resonance. The magnitude is falling off rapidly and the phase is 180 degrees relative to the forcing function. This region of the response curve is completely controlled by the force working to accelerate and decelerate the mass. So at high frequencies, the response is mass controlled.

Figure 4 presents the velocity response for the unit sinusoidal excitation as it is swept from 10 Hz to 100 Hz. Notice the symmetry of the magnitude about the 40 Hz resonant frequency and the asymmetry of the phase about the 0 degree axis at 40 Hz.

Figure 4 : Velocity Response per Newton of Excitation



The velocity response will be very important when considering the acoustic impedance of transmission line geometries. Acoustic equations and calculated results are typically stated in terms of velocity and pressure. Straying slightly off topic for a moment, the acoustic impedance of a transmission line is defined by the following expression.

$$Z_{acoustic}(\omega) = p(\omega) / U(\omega)$$

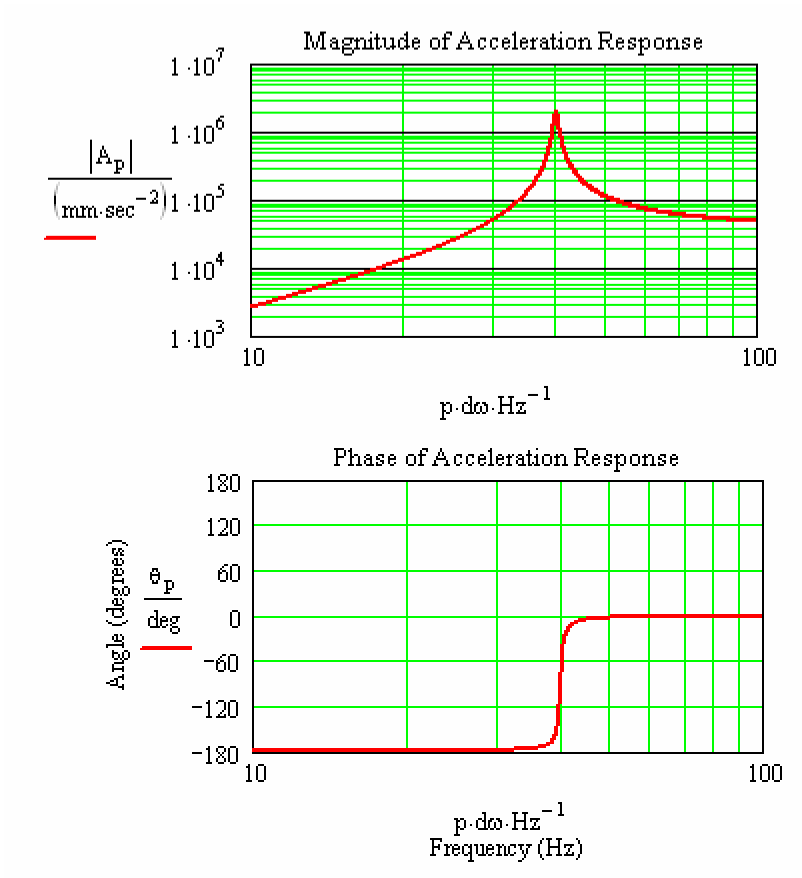
where

$p(\omega)$ = pressure = force / area
$U(\omega)$ = volume velocity = area x velocity

Looking at the velocity phase plot, below resonance the phase is -90 degrees indicating a displacement-controlled response. Above resonance the phase is +90 degrees, which is consistent with an acceleration-controlled response Please review Figure 1 if this is not clear. Finally at resonance, the phase passes through zero confirming that damping is the dominant term at this frequency. These observations parallel the results from the discussion of the displacement response on the previous pages.

Figure 5 shows the acceleration response for the unit sinusoidal excitation as it is swept from 10 Hz to 100 Hz. The magnitude plot looks like a mirror image of the displacement plot while the phase is shifted 180 degrees. Based on the discussion of the displacement and velocity response plots this should not be too surprising. Reviewing the three phase plots, the -90 degree phase shift of the entire plot with each successive differentiation can easily be observed.

Figure 5 : Acceleration Response per Newton of Excitation

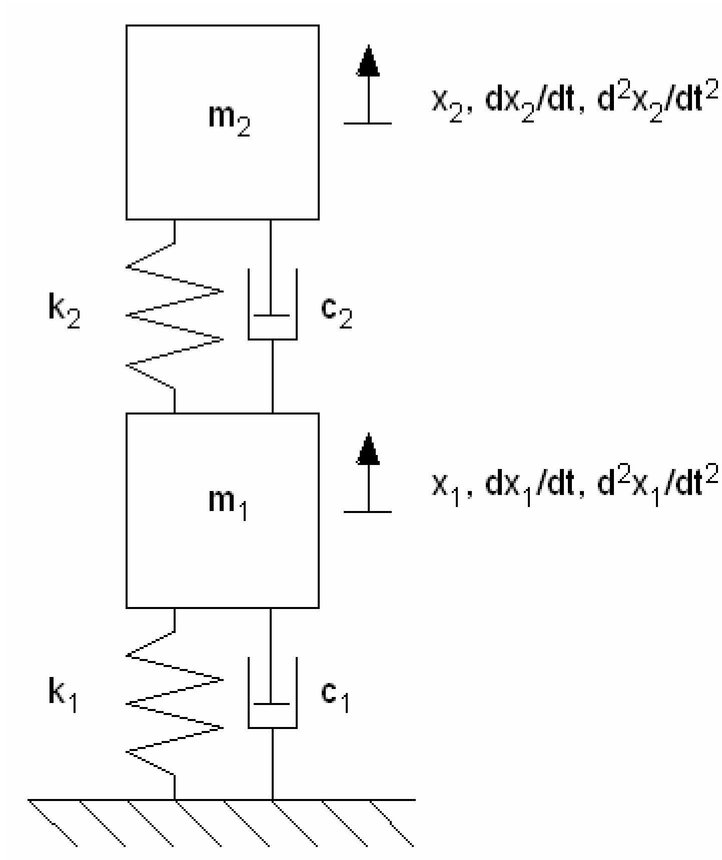


Looking at the acceleration's phase plot, below resonance the phase is -180 degrees which is symptomatic of a displacement-controlled (stiffness controlled) response. Above resonance the phase is 0 degrees, which is again consistent with an acceleration-controlled (mass controlled) response. Finally at resonance, the phase passes through -90 degrees indicating that damping is the dominant term. Once again, these observations parallel the results from the discussion of the displacement and velocity responses on the previous pages.

Two Degree of Freedom System :

If the single degree of freedom system shown in Figure 2 was duplicated, and the pair attached in series, a two degree of freedom system would be formed. Figure 6 shows the resulting system. As described earlier, since there are now two masses we should expect two resonant frequencies. The positive sign conventions are denoted on the right side of the figure by the arrows pointing upward.

Figure 6 : Two Degree of Freedom Mechanical System



The equations of motion for the two degree of freedom system are shown below. Notice that the equations of motion contain acceleration, velocity, and displacement terms reacting forcing functions typically shown on the right side of the equations. For a unit sinusoidal forcing function, F is equal to 1 Newton. In this example, the forcing function is only applied to the lower mass m_1.

$$m_1\, d^2x_1/dt^2 + c_1\, d_1x/dt + k_1\, x_1 - c_2\, (d_2x/dt - d_1x/dt) - k_2\, (x_2 - x_1) = F \cos(\omega\, t)$$

$$m_2\, d^2x_2/dt^2 + c_2\, (d_2x/dt - d_1x/dt) + k_2\, (x_2 - x_1) = 0$$

If the two masses, the two springs, and the two dampers are equal ($m_1 = m_2$, $c_1 = c_2$, $k_1 = k_2$) to the single degree of freedom values; should we expect that the resulting two resonant frequencies are both 40 Hz? Again, an undergraduate level vibration textbook will contain the detailed derivation and solution of the coupled differential equations of motion for the two degree of freedom system, so it will not be repeated. Assuming this is a lightly damped system (high Q), Blevins' textbook(4) can be consulted. This valuable textbook contains a comprehensive set of tables containing equations for the natural frequencies and corresponding mode shapes of a wide variety of physical systems.

If we ignore damping for the moment, the equations shown in Table 6-2, Case 2, on page 48 can be used to calculate the natural frequencies of the two degree of freedom system to be calculated.

$$f_1 = 0.618\ [1/2\pi\ (k / m)^{1/2}] = 0.618\ (40\ \text{Hz}) = 24.7\ \text{Hz}$$

$$f_2 = 1.618\ [1/2\pi\ (k / m)^{1/2}] = 1.618\ (40\ \text{Hz}) = 64.7\ \text{Hz}$$

Also from the same table, the mode shapes can be calculated normalized to the first mass m_1. The normalized mode shape displacements are shown below.

$$[X_1 , X_2] = [1.0 , 1.618]$$

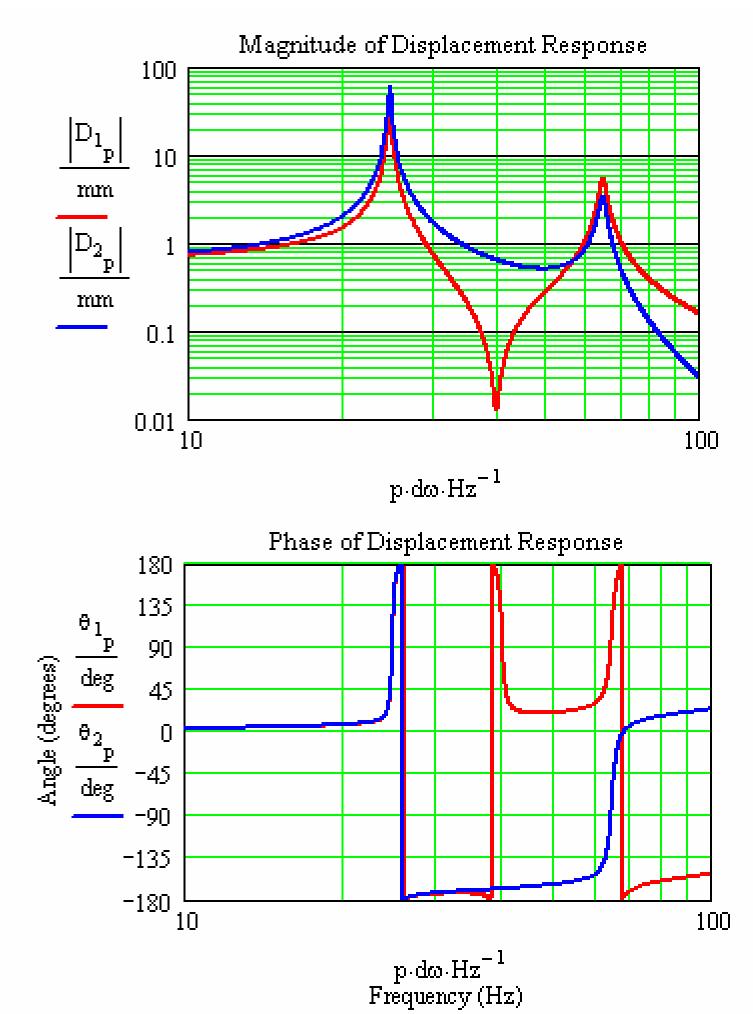
$$[X_1 , X_2] = [1.0 , -0.618]$$

The concept of natural frequencies and mode shapes for multi-degree of freedom systems could probably use a few more sentences of explanation. For the two degree of freedom system shown in Figure 6, two resonant frequencies and two mode shapes were calculated. Neither of the two resonant frequencies corresponds to the 40 Hz resonance of the individual single degree of freedom systems which were combined to form this example. At these two resonant frequencies, a significantly amplified response should be expected similar to the single degree of freedom system described previously. If the system were excited at one of these natural frequencies, the amplitudes and phases of the motions of the two masses would be proportional to the matching mode shape defined above.

Some properties of the mode shapes also need to be stated. Since the damping has not been included in the closed form solutions used to generate Blevins' tables, the exact amplitude of the vibratory motion cannot be uniquely determined. Mode shapes can be specified normalized to unity at one location, in this case the mass m_1 where the forcing function is being applied. Since this is a linear system, as discussed earlier in this article, the absolute amplitude at the resonant frequency can be defined by the mode shape and an appropriate scale factor. Any vibrating deflected shape, at any arbitrary frequency, can be represented as a linear combination of a system's mode shapes. The mode shapes become a form of coordinate system in the same way that an arbitrary position in space can be completely specified by three coordinates, for example the x, y, or z Cartesian coordinates.

Once again, this is a very lightly damped mechanical system that can be subjected to a unit sinusoidal excitation and solved numerically at discrete frequencies using MathCad. Figures 7, 8, and 9 show the displacement, velocity, and acceleration responses for this two degree of freedom system when subjected to a 1 Newton sinusoidal excitation applied to mass m_1. The red curve in each of the plots represents the response of mass m_1 while the blue curve represents the response of mass m_2.

Figure 7 : Displacement Responses per Newton of Excitation



The most obvious feature of the displacement response plot, shown in Figure 7, is the two separate resonance peaks. The frequencies, magnitudes at the peaks, and the relative phase of the two masses were extracted from the displacement response plot and are shown below.

f_1 = 24.7 Hz and [X_1 , X_2] = [36.768 , 59.422] in phase

f_2 = 64.7 Hz and [X_1 , X_2] = [5.398, 3.341] out of phase

The peak magnitudes can both be normalized, relative to m_1, and the sign of the m_2 displacement adjusted to account for the in or out of phase condition.

$$f_1 = 24.7 \text{ Hz and } [X_1 , X_2] = [1.0 , 1.616]$$

$$f_2 = 64.7 \text{ Hz and } [X_1 , X_2] = [1.0 , -0.619]$$

These results match very closely the predictions from the tables contained in Blevins' textbook. The results above are the two resonant frequencies and the two normalized mode shapes for this particular two degree of freedom system.

The other interesting feature in the plot is the deep narrow null in the displacement curve of m_1 at the single degree of freedom resonant frequency of 40 Hz. At 40 Hz, the lower mass m_1 essentially stops moving while the motion of m_2 continues but at a reduced amplitude.

As stated earlier, any particular set of displacements can be expressed as a scaled linear combination of the two mode shapes. For example, consider the following scaled combination.

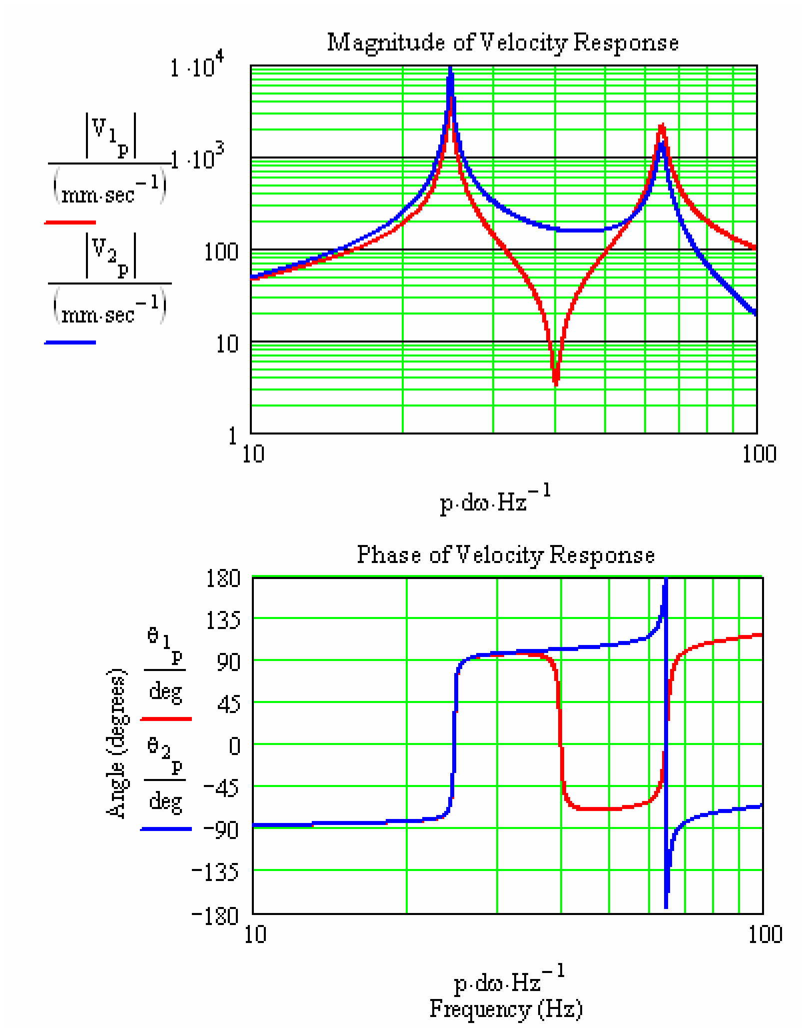
$$0.3 \{ -[1.0 , 1.616] + [1.0 , -0.619] \} = [0.0 , -0.671]$$

These values are essentially equal to the values extracted from Figure 7 at 40 Hz. The negative sign indicates that m_2 is vibrating out of phase with the forcing function. The actual calculated results, taken from Figure 7 at 40 Hz, are [0.013 , -0.633].

One last characteristic of the displacement plot is the slope of the curves, above the second resonance, at the extreme right of the plot. The negative slope is double the rate of the negative slope seen in the single degree of freedom displacement plot. For the single degree of freedom system the slope is proportional to $1/\omega$ while for the two degree of freedom system the slope is proportional to $1/\omega^2$.

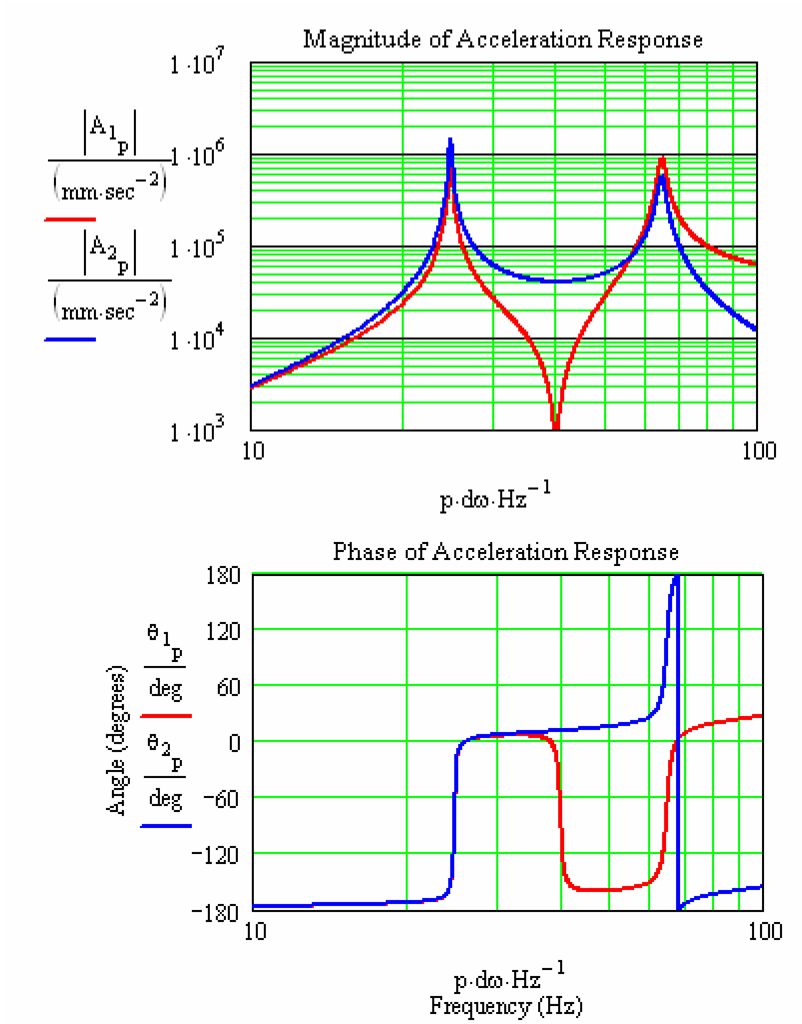
Another interesting property of the two degree of freedom response curves is probably most easily seen in the velocity response shown in Figure 8. Looking specifically at the response for m_1, the red curve, if you split the plot at 40 Hz the two pieces would strongly resemble separate single degree of freedom system plots similar to the curve shown in Figure 4. The first single degree of freedom system has a resonant frequency at 24.7 Hz while the second single degree of freedom system has a resonant frequency at 64.7 Hz. Both systems start with a phase of -90 degrees and transition to a phase of +90 degrees above each resonance. In the middle, at 40 Hz, the two single degree of freedom plots are approximately equal in magnitude and 180 degrees out of phase resulting in the deep null.

Figure 8 : Velocity Responses per Newton of Excitation



The acceleration response, contained in Figure 9, shows similar trends and produces many of the same observations as the displacement and velocity plots. Again, the acceleration response magnitude plot is very similar to a reversed displacement magnitude plot. In the displacement plot, the falling slope above the second resonance was twice as steep as the single degree of freedom plot. Now in the acceleration plot, the rising slope below the first resonance is twice as steep as was observed in the corresponding single degree of freedom plot. For the single degree of freedom system the slope is proportional to ω while for the two degree of freedom system the slope is proportional to ω^2. If we relate these acceleration slopes to speaker design, a closed box system (single degree of freedom) has a 12 dB/octave slope below the tuning frequency while a ported box system (two degrees of freedom) has a 24 dB/octave slope below the box tuning frequency.

Figure 9 : Acceleration Responses per Newton of Excitation



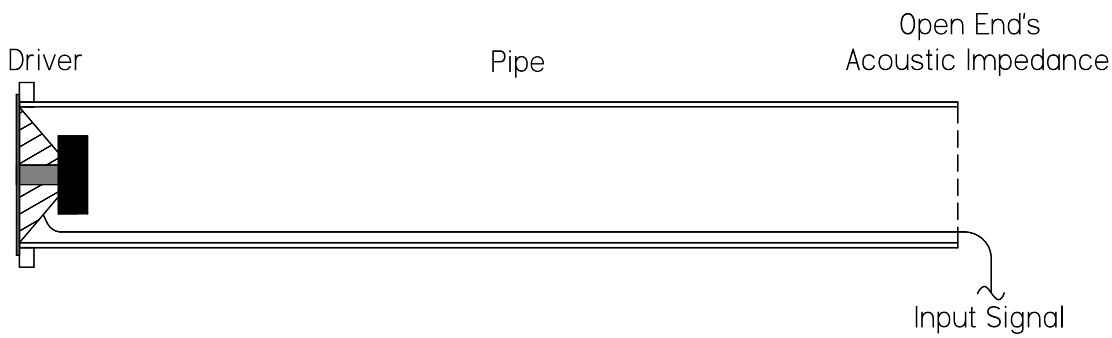
Extending the discussion, and becoming a little more abstract, the response of the two degree of freedom system can be thought of as a summed response of two new single degree of freedom systems with resonant frequencies at 24.7 and 64.7 Hz. These two new single degree of freedom systems are not simply the two original single degree of freedom systems, with masses m_1 and m_2 and 40 Hz natural frequencies, but are based on the mass and stiffness distribution of the assembled system. The vibration of the assembled system is defined by discrete frequencies and mode shapes. Each mode shape is unique and independent of the others. Any motion of the assembled mechanical system can be represented as a linear combination of these mode shapes. An analogy that might help people visualize this situation is the representation of a square wave as a linear summation of sine waves of different periods and magnitudes. The different periods represent unique frequencies and the resulting sine waves could be thought of as the mode shapes. The square wave can be expressed as the sum of an infinite number of sine waves.

In summary, a few key concepts were illustrated in this section for lightly damped mechanical systems (hopefully clearly enough to be understood). First, two single degree of freedom systems were combined to form a two degree of freedom system. The original independent single degree of freedom resonant frequencies shift to become two new resonance frequencies that bracket the original values. This is a common phenomenon seen when bringing two independent dynamic systems together to form a larger combined dynamic system. Please recognize that the amount of frequency shifting will be a function of the physical properties of the independent systems. Second, the concept of multiple system natural frequencies and mode shapes was shown and it was demonstrated that any motion can be expressed as a scaled linear combination of the mode shapes. And finally, the relationship between the phase of displacement, velocity, and acceleration was shown and observed in the different response plots above and below resonance and as a 180 degree phase shift that occurs when passing through a system resonance.

The Classic Straight Transmission Line Loudspeaker System :

The transmission line loudspeaker system is the combination of two independent dynamic systems, the driver and the transmission line enclosure. Figure 10 shows a hypothetical transmission line loudspeaker system. The three key components in the transmission line loudspeaker are the driver, the enclosure pipe, and the open end's acoustic impedance. These three features determine the fundamental vibration characteristics, the resonant frequencies and mode shapes, of the transmission line loudspeaker system.

Figure 10 : A Transmission Line Loudspeaker System



What about the effect of the fiber stuffing? This question is probably already troubling some readers. The fiber stuffing in a transmission line loudspeaker has been the black art in the design process for decades. The true role of the fiber stuffing has only recently started to be understood and described mathematically. To fully understand the vibration properties of the transmission line loudspeaker, the empty lightly damped system needs to be simulated and studied first. Adding fiber stuffing to a transmission line significantly attenuates the higher order standing waves but does not fundamentally change the basic physics; the fiber stuffing tends to hide the physics from view. Therefore, the following discussion will concentrate on an empty transmission line enclosure. The last step will be to introduce the fiber stuffing and demonstrate the performance benefits that it brings to the system.

The starting point for the discussion is the transmission line enclosure itself. Analyzing this system will allow the basic relationships between resonant frequencies and mode shapes to be established. This is the most consistent discussion with what appears in classic acoustics texts and with most people's basic understanding of transmission line physics. Next the open end's acoustic impedance will be shown and the influence added to the classic acoustics results for the transmission line enclosure. This is a big step that is not really documented well in acoustics texts and is probably new ground for most readers. Then the independent driver and the transmission line enclosure systems will be assembled and the resulting combined system's vibration properties plotted and discussed.

The Classic Acoustic Pipe Problem :

The problem presented and solved in most acoustics texts[5, 6] is the one dimensional wave equation applied to a straight pipe where one end is open and the other end is excited by a rigid oscillating piston. The partial differential equation of motion is shown below

$$\partial^2 u(x,t)/\partial t^2 = c^2\ \partial^2 u(x,t)/\partial x^2$$

with the assumed boundary conditions

$$u(0,t) = u_d\ e^{j\omega t}$$
$$\partial u(L,t)/\partial x = 0$$

The relationship between pressure and velocity is used to completely describe the dynamics inside the pipe and characterize the impedance as seen by the oscillating piston.

$$-\partial p(x,t)/\partial x = \rho\ \partial u(x,t)/\partial t$$

The equation of motion can be solved along the entire length of the pipe. The details of the solution can be found in the acoustics textbooks. Skipping right to the solution for the velocity and the pressure yields the following two equations.

$$u(x,t) = [u_d \tan(\omega/c\ L) \sin(\omega/c\ x)+u_d \cos(\omega/c\ x)]\ e^{j\omega t}$$
$$p(x,t) = j\ \rho c\ [u_d \tan(\omega/c\ L) \cos(\omega/c\ x) +u_d \sin(\omega/c\ x)]\ e^{j\omega t}$$

where

j = imaginary number
$\omega = 2\ \pi\ f$
c = speed of sound, typically about 343 m/sec
ρ = density of air, typically about 1.21 kg/m^3
L = length of the pipe

To characterize the pipe, an expression for the acoustic impedance is derived. At x = 0, the velocity and pressure can be determined, from the equations above, and used to calculate the acoustic impedance.

$$Z_{acousitc}(\omega) = p(0) / [S_0\ u(0)]$$

$$Z_{acousitc}(\omega) = j\ \rho c \tan(\omega/c\ L) / S_0$$

For a transmission line, the cross-sectional area is denoted by the variable S with a subscript 0 or L to indicate if it is defined at the closed end (x = 0) or the open end (x = L) respectively. The pipe shown in Figure 10 has a constant cross-sectional area so $S = S_0 = S_L$. From the expression for the acoustic impedance, it can be shown that at specific frequencies, the impedance becomes infinite, since $\tan(n\ \pi/2) = +\infty$ or $-\infty$ when n = 1, 3, 5, ….. These frequencies are the quarter wavelength resonant frequencies of the straight pipe and occur at evenly spaced and predictable values.

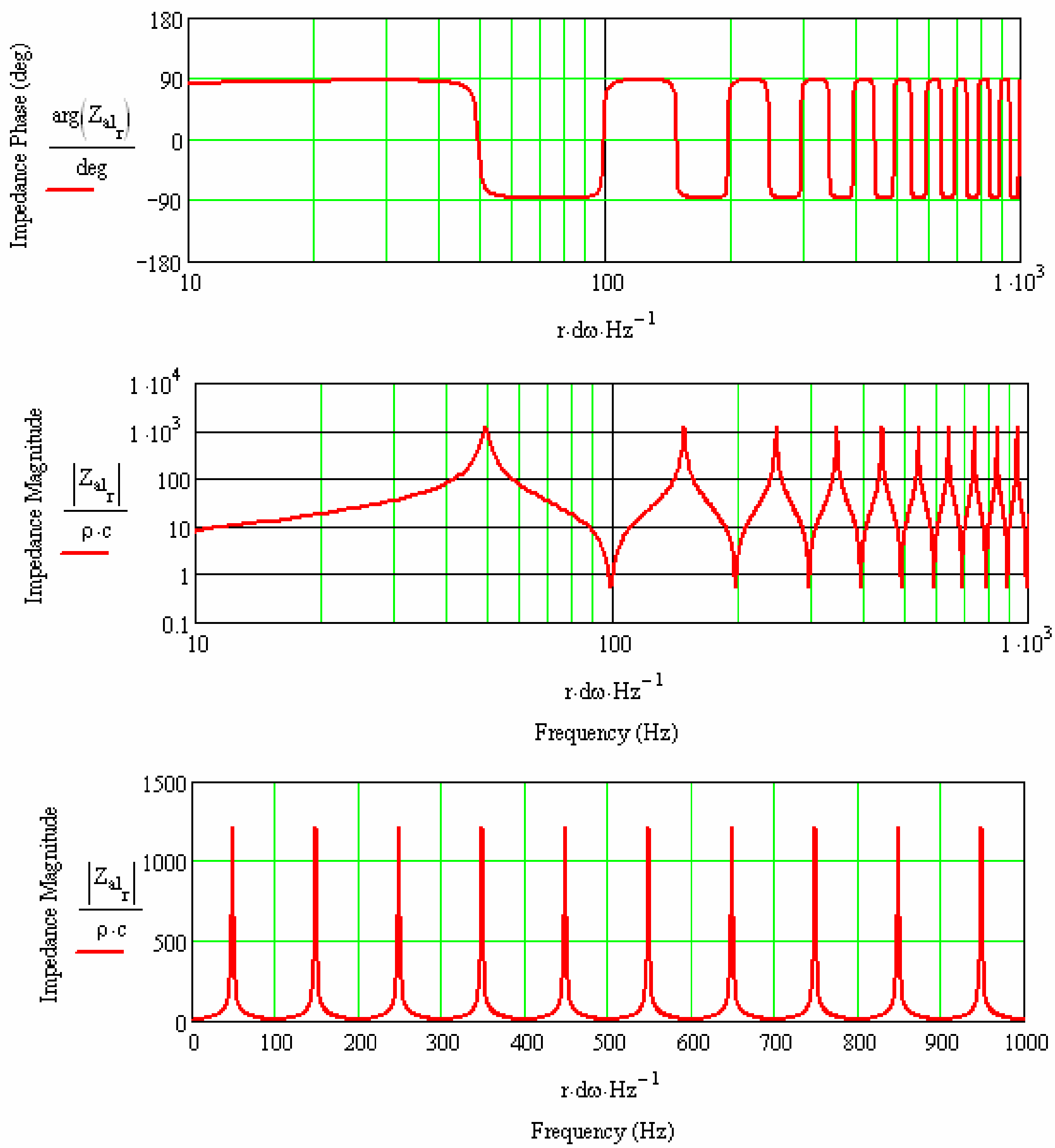
$f_n = n\ c\ /\ (4\ L)$ where $n = 1, 3, 5, 7, 9$ ……

To demonstrate the natural frequencies and mode shapes of a classic open ended transmission line, a numerical example will be used to plot the responses. Assuming that the desired tuning frequency is 50 Hz, the required length can be calculated.

$$L = n\ c\ /\ (4\ f_n) = 1 \times (343 \text{ m/sec})\ /\ (4 \times 50 \text{ Hz}) = 1.72 \text{ m} = 67.5 \text{ inches}$$

The cross-sectional area is defined equal to 400 cm^2, approximately 2 x S_d of an eight inch driver. Figure 11 shows the acoustic impedance seen by the oscillating rigid piston.

Figure 11 : Straight Transmission Line Enclosure Acoustic Impedance



The top two plots in Figure 11 contain the magnitude and phase of the acoustic impedance plotted using a logarithmic frequency scale. The bottom plot in Figure 11 reproduces the magnitude plot but using a linear frequency scale. In the bottom plot, the evenly spaced and predictable resonant frequencies are easily seen at 50, 150, 250, … 950 Hz. These are the fundamental 50 Hz quarter wavelength resonance and the first nine odd harmonics.

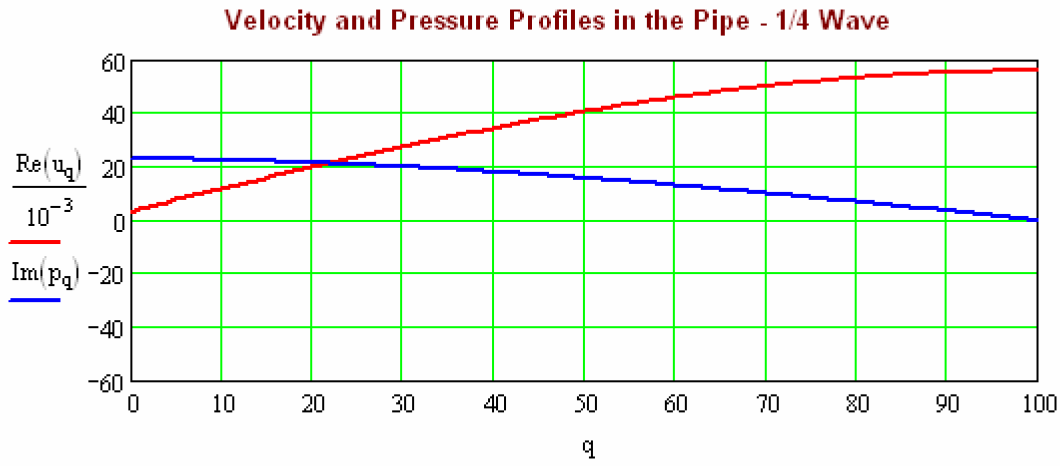
In Figure 12 the first three pairs of standing waves, known as mode shapes, are shown as a function of the percentage of the pipe's length. In each plot the velocity is represented by the red curve and the pressure by the blue curve. There are a number of classic transmission line properties depicted in these curves which are itemized below.

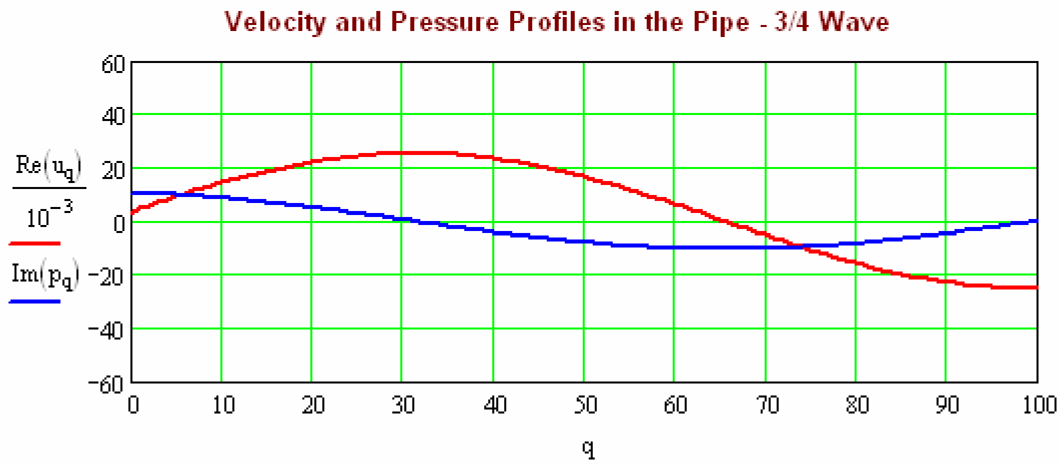
1. The pressure is at a maximum at the closed end and is zero at the open end.
2. The velocity is near a minimum at the closed end, it is equal to the small excitation velocity u_d, and is a maximum at the open end.
3. In the 3/4 wave plot, the pressure passes through zero and the velocity is a maximum at 33% of the length. The acoustic impedance at this point is zero. If a driver were placed at this axial position, the 3/4 wave resonance would not be excited.
4. Since the velocity excitation is a constant, the displacement excitation decreases at higher frequencies. This explains the decreasing magnitudes of the standing waves for the higher harmonics.

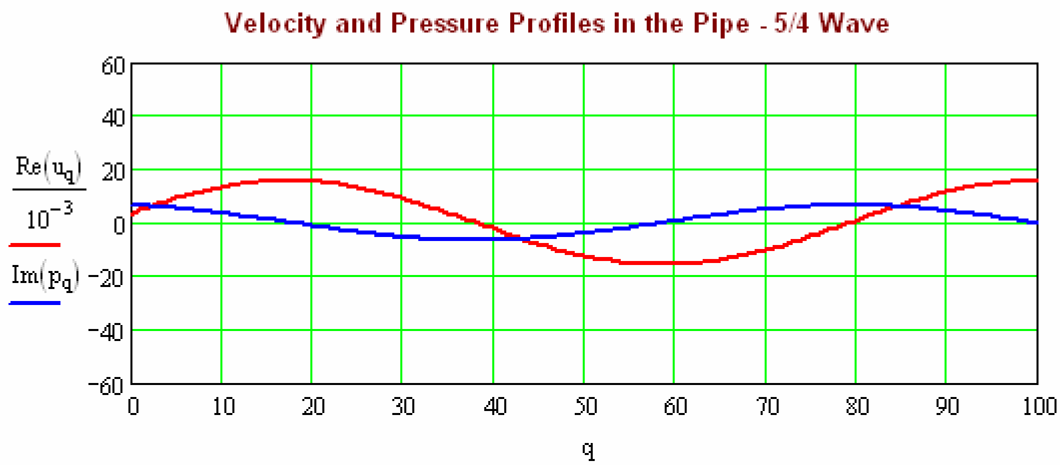
When studying the mode shapes in Figure 12, it should be recognized that these are captured at an instant in time corresponding to the peak magnitudes. In reality every point on the mode shape is vibrating vertically about the x axis (y = 0). The mode shapes could have just as easily been shown as a mirror image below the x axis. If time were moving forward, the lines would oscillate up and down with the same relative shape between the position shown and the mirror image position below the x axis. The rate at which the lines move between these two extreme positions is determined by the resonant frequency, the higher order mode shapes move up and down at a much more rapid pace.

The frequencies of the standing wave resonances and the shapes of the waves, the mode shapes, are probably what most people associate with classic transmission line enclosures. The plots presented so far should be a simplified review for most transmission line aficionados. Unfortunately, as we introduce reality into the classic closed form solution of the wave equation, things will get a little more complicated.

Figure 12 : Quarter Wavelength Standing Waves
$\partial u(L,t)/\partial x = 0$ at $x = L$







The Open End Acoustic Boundary Condition :

The next level of refinement to the transmission line enclosure model is to replace the simple boundary condition at the open end with an acoustic impedance boundary condition. The new boundary conditions are shown below.

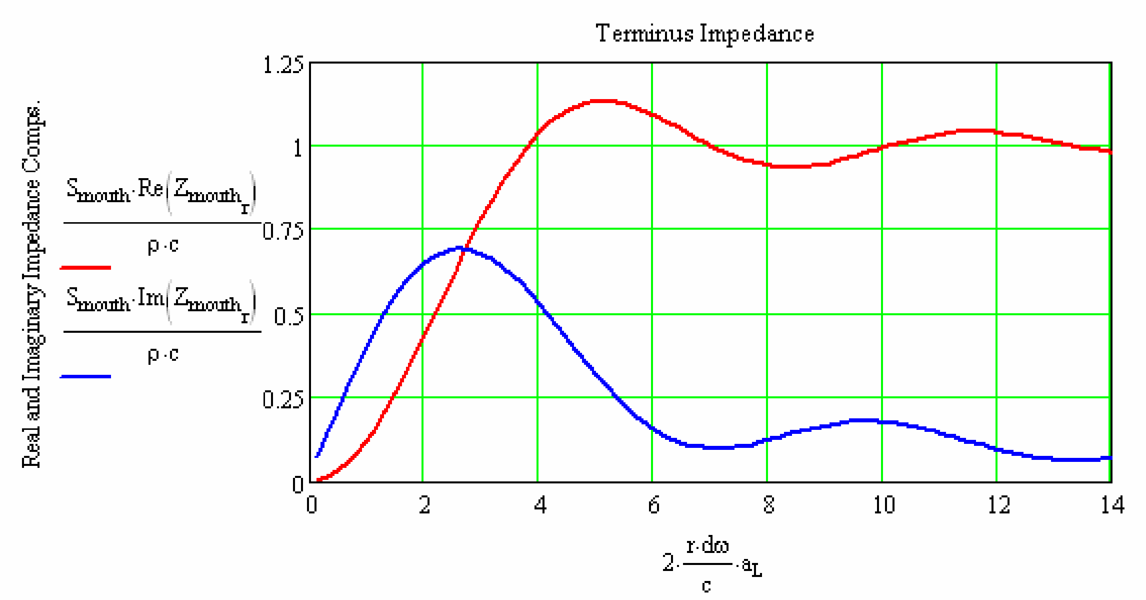
$$u(0,t) = u_d \, e^{j\omega t}$$
$$p(L) / [S_L \, u(L)] = Z_{mouth}(\omega)$$

The boundary condition at the open end of a transmission line is modeled using the acoustic impedance of a piston in an infinite baffle. The equation for the acoustic impedance of a circular piston in an infinite baffle is shown below and plotted in Figure 13.

$$Z_{mouth}(\omega) = \rho c / S_L \{[1 - 2\, J_1(2\, \omega/c\, a_L) / (2\, \omega/c\, a_L)] + j\, [2\, H_1(2\, \omega/c\, a_L) / (2\, \omega/c\, a_L)]\}$$

a_L = radius of the equivalent circular piston
J_1 and H_1 = forms of Bessel functions
$S_L = S_{mouth}$

Figure 13 : Assumed Acoustic Impedance Boundary Condition
at the Open End of a Transmission Line
Red Curve = Real Part
Blue Curve = Imaginary Part



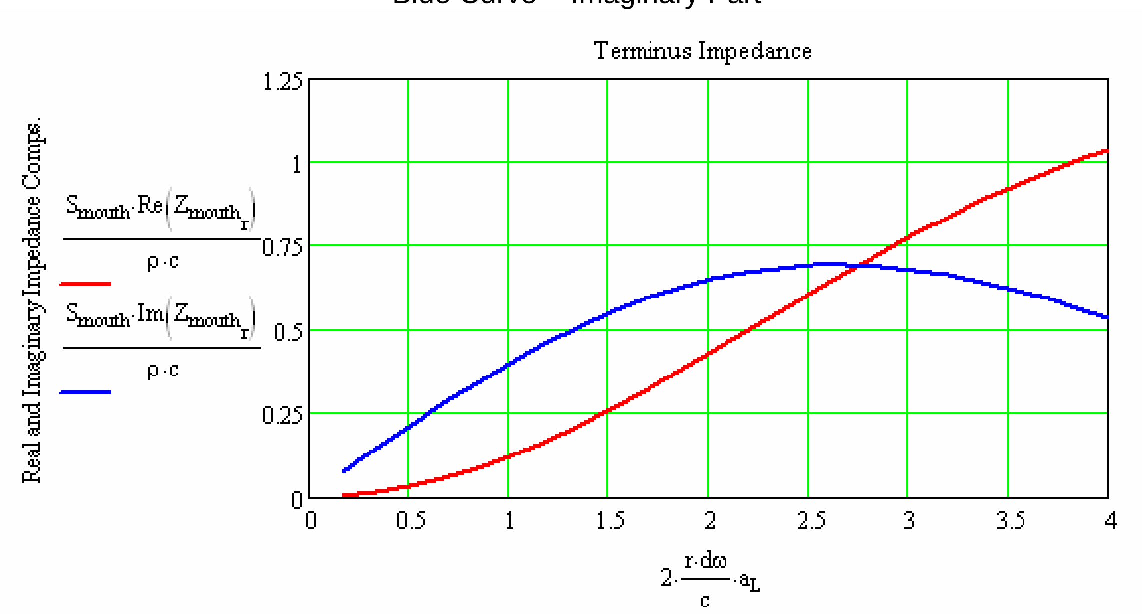
The radius of an equivalent piston can be calculated from a rectangular cross-section. Since we are interested in frequencies below 1000 Hz, the position along the curve in Figure 13 can be defined.

$$a_L = (400 \text{ cm}^2 / \pi)^{1/2} = 11.3 \text{ cm} = 4.44 \text{ inches}$$

$$2\ \omega/c\ a_L = 2\ (1000\ \text{Hz} / 2 / \pi / 343\ \text{m/sec})\ 0.113\ \text{m} = 4.146$$

Re plotting Figure 13 to show only the frequency range of interest produces Figure 14. The real part of the acoustic impedance, the red curve, acts as a damper transferring energy from the open end of the transmission line into the room. The imaginary part of the acoustic impedance, the blue curve, acts as an additional mass reflecting energy back into the transmission line and also making the line effectively a bit longer then the previously calculated actual physical length. The reflection of energy back into the transmission line is what produces standing waves.

Figure 14 : Acoustic Impedance at the Open End
Red Curve = Real Part
Blue Curve = Imaginary Part



The impact of adding the open end acoustic impedance to the mathematical model of the transmission line enclosure can be seen in the impedance plots shown in Figure 15. Comparing Figures 11 and 15, the influence of the real part of the open end acoustic impedance is seen as a dramatic attenuation of the higher order standing waves. Also, if you look closely the imaginary part has caused the tuning frequency to drop slightly in Figure 15, compared to Figure 11, for the same physical length of transmission line enclosure. The new slightly lower resonant frequencies are at 47, 142, 238, … 920 Hz. Figure 16 presents the same quarter wave mode shapes, that were shown in Figure 12, including the new boundary condition at the open end.

Figure 15 : Straight Transmission Line Enclosure Acoustic Impedance
Impedance Boundary Condition at x = L

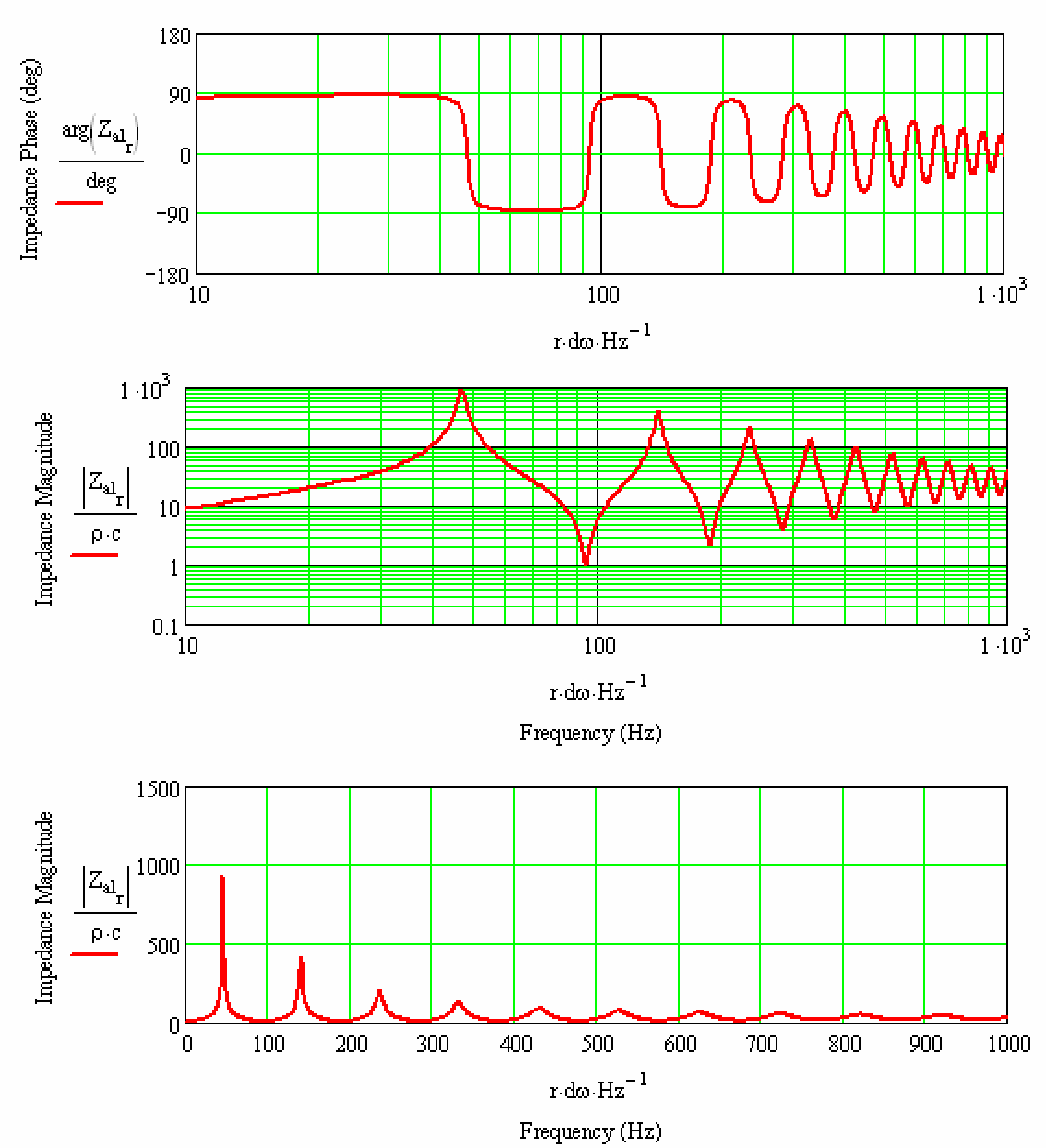
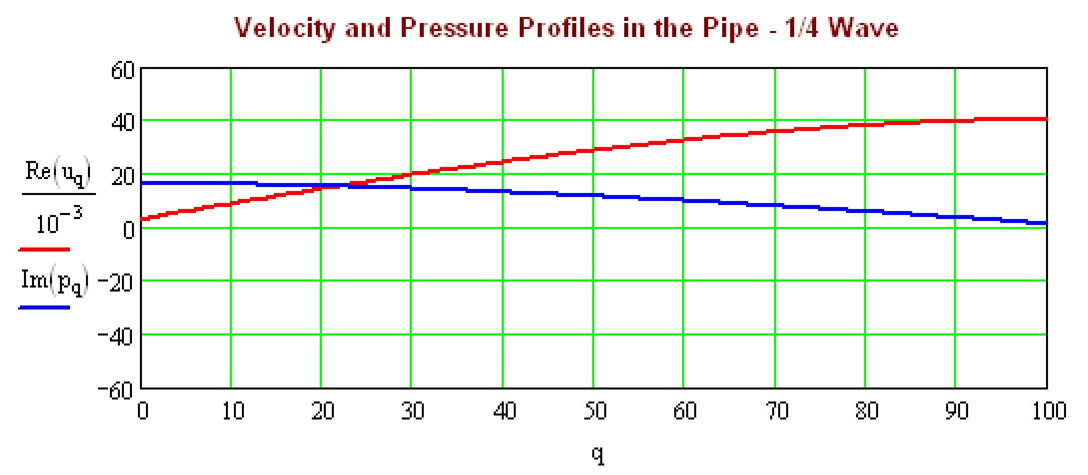
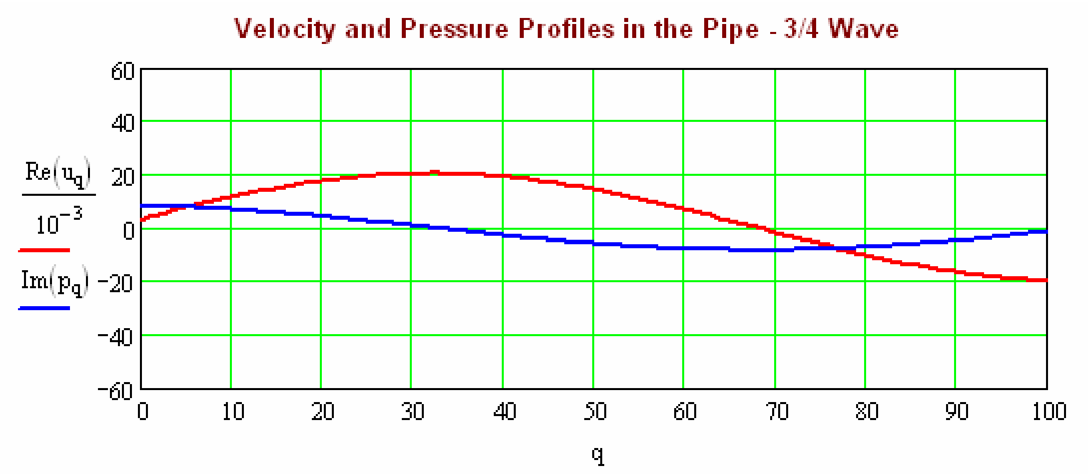
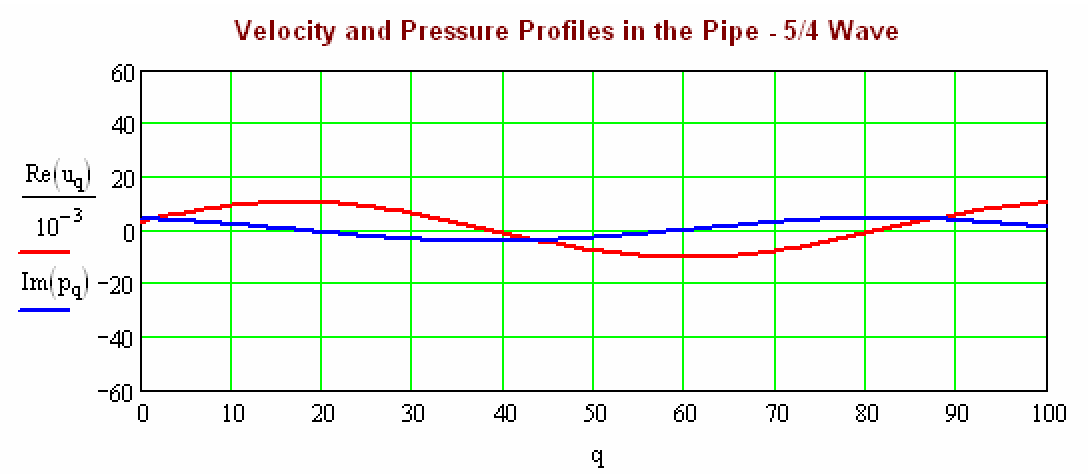


Figure 16 : Quarter Wavelength Standing Waves
Impedance Boundary Condition at x = L







At this point in the discussion, the model of the transmission line enclosure and its more complex open end boundary condition has been demonstrated. The key points to be remembered are that the straight pipe has resonant frequencies which are reasonably predictable, that the mode shapes are odd integer multiples of sine (velocity) and cosine (pressure) quarter waves, and that the open end boundary condition significantly attenuates the higher order modes. This model represents one of the two physical subsystems that need to be combined to form a complete transmission line speaker system.

The Driver :

The assumed driver will also have a resonant frequency of 50 Hz to match the calculated physical length of the transmission line enclosure. A generic driver is defined below based on key input properties and resulting derived properties. When looking at the relationships used to calculate the derived properties, please keep in mind that MathCad internally automatically converts frequency in Hz to frequency in rad/sec. This property of MathCad leads to equations that may not look exactly like those familiar to the DIY speaker designer. In equations containing a frequency term, a 2π multiplier may be missing or added depending on the desired units of the calculated result. The electrical impedance of the generic driver in an infinite baffle is plotted in Figure 17; the resonance generated peak is clearly seen at 50 Hz.

Driver Thiele / Small Parameters : Generic Driver Derivation

$f_d := 50 \cdot Hz$ $\qquad Q_{md} := 6$

$R_e := 8 \cdot ohm$ $\qquad Q_{td} := 0.4$

$L_{vc} := 0 \cdot mH$ $\qquad SPL := 90$ dB

$S_d := 200 \cdot cm^2$

Derived Thiele / Small Parameters

$Q_{ed} := \left(\frac{1}{Q_{td}} - \frac{1}{Q_{md}}\right)^{-1}$ $\qquad Q_{ed} = 0.429$

$\eta_o := 10^{\frac{SPL-112}{10}}$ $\qquad \eta_o = 0.631\,\%$

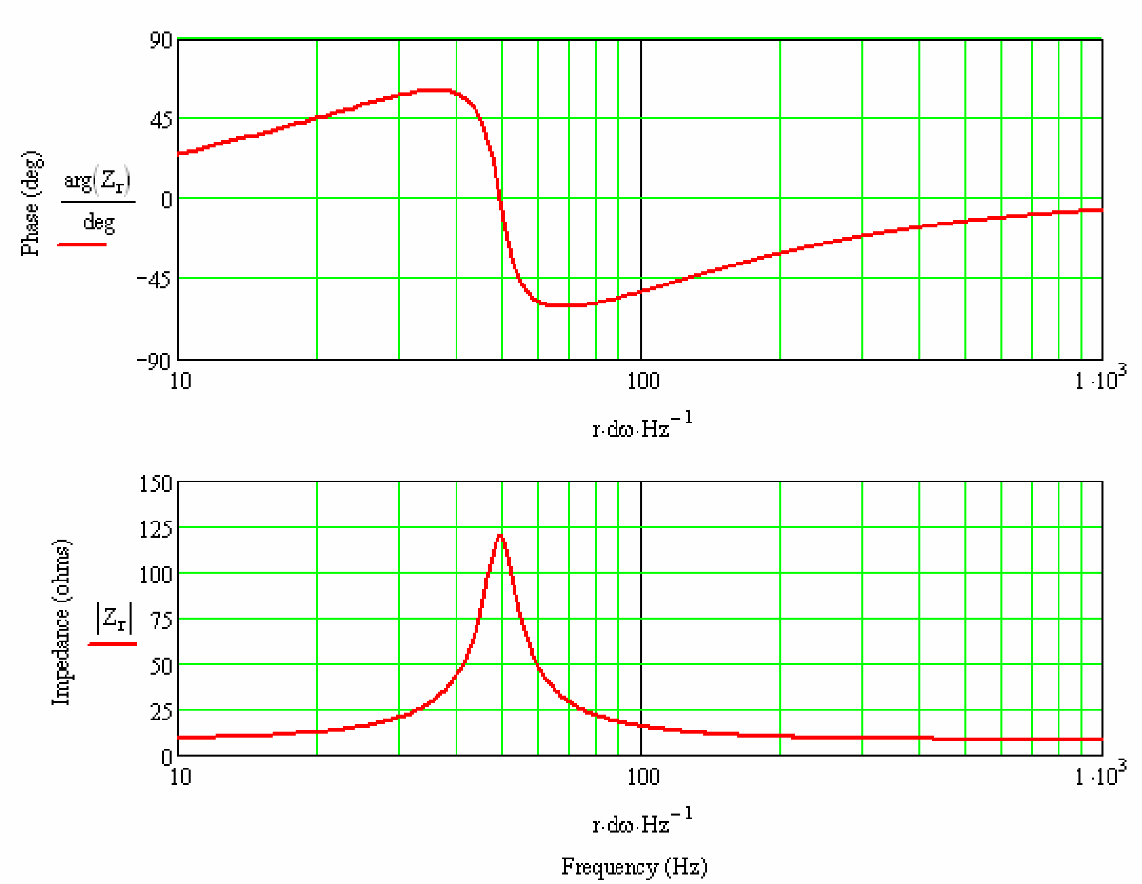
$V_d := \eta_o \cdot 2 \cdot \pi \cdot c^3 \cdot Q_{ed} \cdot f_d^{-3}$ $\qquad V_d = 21.920\,liter$

$C_{md} := V_d \cdot \left(\rho \cdot c^2 \cdot S_d^2\right)^{-1}$ $\qquad C_{md} = 3.872 \times 10^{-4} \frac{m}{newton}$

$M_{md} := \left(C_{md} \cdot f_d^2\right)^{-1}$ $\qquad M_{md} = 26.168\,gm$

$Bl := \left(\frac{f_d \cdot R_e \cdot M_{md}}{Q_{ed}}\right)^{0.5}$ $\qquad Bl = 12.388 \frac{newton}{amp}$

Figure 17 : Electrical Impedance of the Generic Driver Mounted in an Infinite Baffle



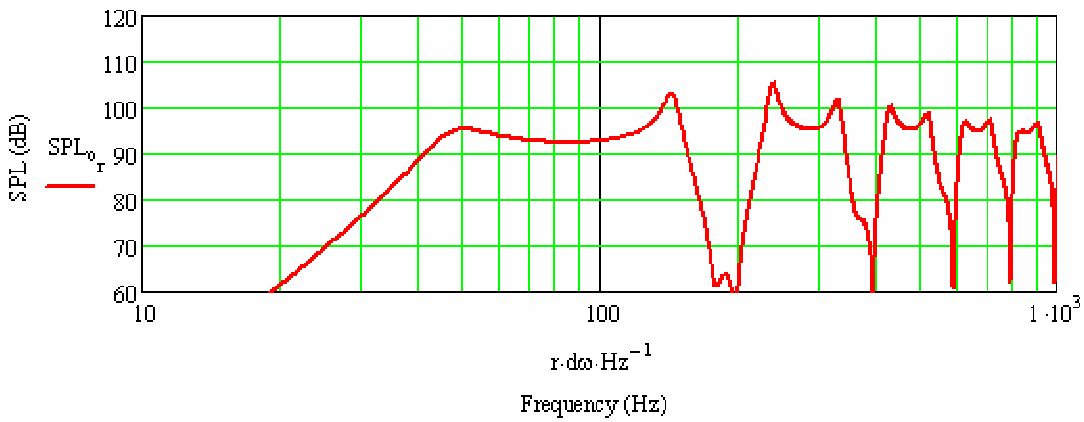
Combining the Driver with the Transmission Line Enclosure to Form a System :

At this point in the discussion, we have a driver which is a single degree of freedom system and a transmission line enclosure which is a multi-degree of freedom system. Both systems have fundamental tuning frequencies of approximately 50 Hz. Remembering what happened previously when we combined two single degree of freedom systems, we should expect that the combined transmission line speaker system will have a pair of resonant frequencies that bracket 50 Hz. To demonstrate this theory, the driver was placed at the closed end of the unstuffed transmission line enclosure and the MathCad worksheet “TL_Open_End.mcd” was used to perform a computer simulation. The SPL curves and the electrical impedance curve were calculated and the results are shown in Figure 18.

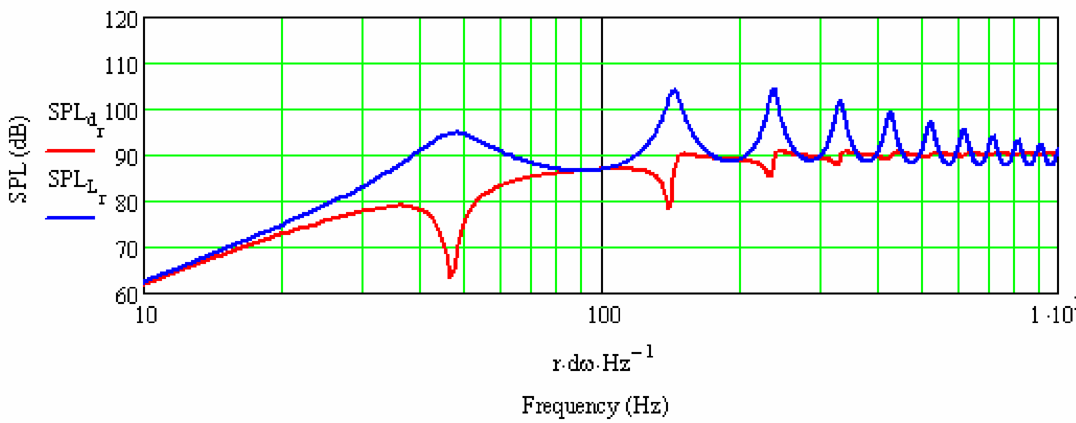
The top plot in Figure 18 is the combined system SPL response. Without fiber damping, the response is extremely ragged above 200 Hz due to the strong enclosure resonances. However, below 200 Hz the response is not too bad with a 24 dB/octave roll-off below 50 Hz. Adding the fiber damping in the next section will greatly improve the overall SPL response by significantly reducing the deep nulls to become more of a rippled response.

Figure 18 : Straight Transmission Line System Simulation Results
(Stuffing Density = 0.0 lb/ft^3)

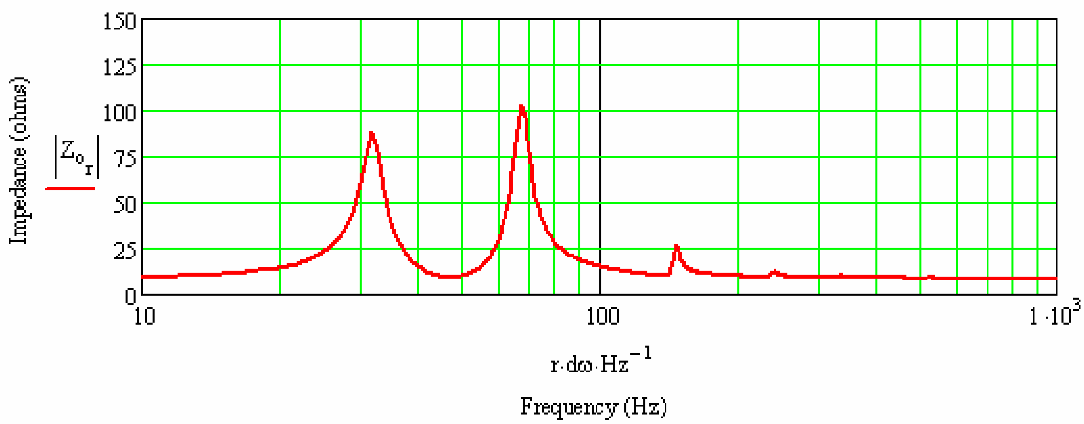
Far Field Transmission Line System Sound Pressure Level Response



Woofer (red curve) and Open End (blue curve) Far Field Sound Pressure Level Responses



Transmission Line System Electrical Impedance



The bottom two plots in Figure 18 seem to present a conflicting story with respect to the resonances in the transmission line speaker system. The first three peaks in the open end's response, shown in the middle plot, occur at 48, 146, and 240 Hz which are almost the same as the transmission line enclosure results shown in Figure 15. However, the electrical impedance plot at the bottom of Figure 18 has peaks at 32, 68, 149, and 242 Hz with a minimum near 50 Hz. To try and determine which peaks represent system resonances, and which are not, the pressure and velocity profiles along the length of the transmission line are plotted in Figure 19.

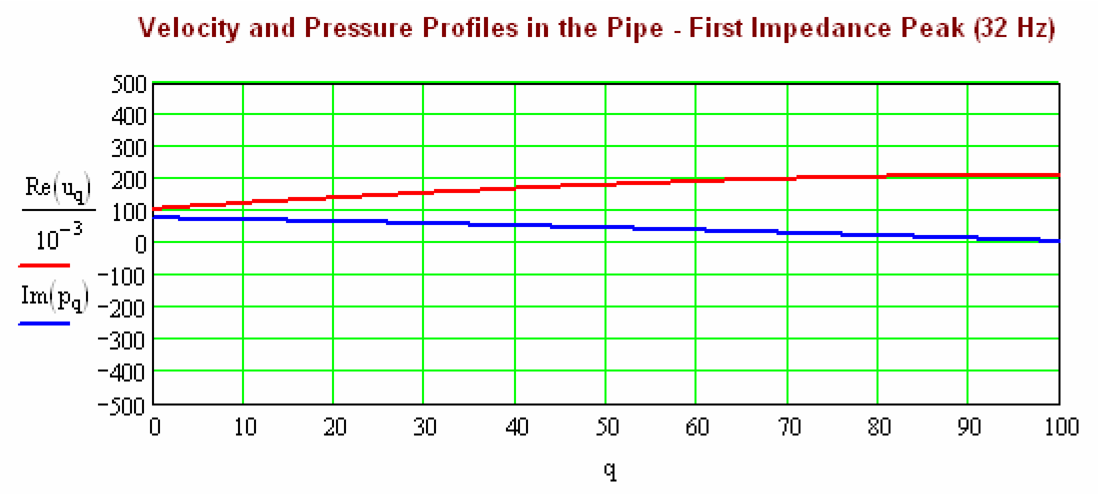
Starting with the higher frequency mode shapes at 149 and 242 Hz, the mode shapes look very similar to the lower two plots in Figure 16. The frequencies and the mode shapes are almost identical to the transmission line enclosure results. Adding the driver to the transmission line enclosure had a small impact on these higher frequency modes.

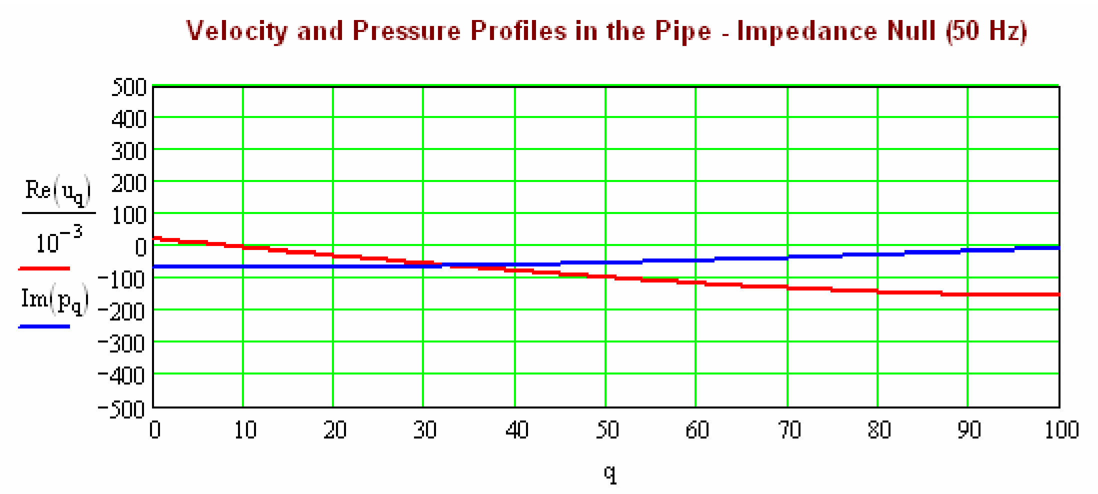
The top plot in Figure 16 looks similar to the middle plot in Figure 19. However, think back to the two degree of freedom problem where the resonant frequencies were the two peaks in the displacement plot and the null in the middle was just a linear combination of the two mode shapes. Assume that the resonant frequencies and mode shapes correspond to the results at 32 and 68 Hz which are the two electrical impedance peaks in Figure 18. Earlier it was stated that the mode shapes form a coordinate system and that any motion can be expressed as a scaled linear combination of the mode shapes. Scaling the plots corresponding to the electrical impedance peaks, in Figure 19, and then adding them produces the velocity and pressure curves shown in Figure 20. The curves shown in Figure 20 match the velocity and pressure profiles at the electrical impedance minimum as seen in the second plot of Figure 19.

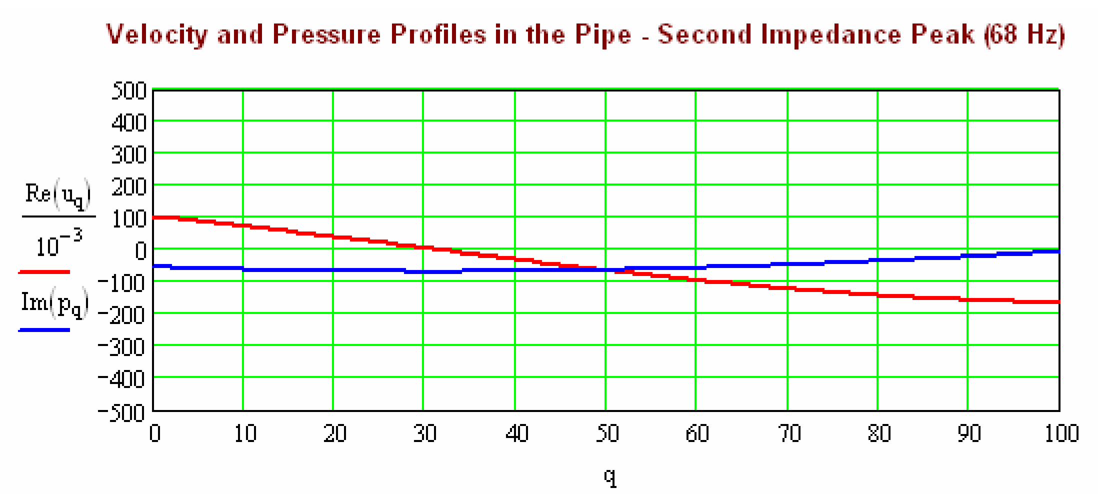
Reversing the argument, assume that 48 Hz is a resonance and the mode shapes are the quarter waves shown in the second plot of Figure 19. Can this velocity and pressure profile be scaled and added to any of the other mode shapes to produce the velocity and pressure profiles contained in the first and third plots of Figure 19? The answer is no, therefore 50 Hz is not a resonant frequency and the velocity and pressure profiles are not mode shapes.

In summary, the first four resonant frequencies and mode shapes for the transmission line speaker system occur at 32, 68, 149, and 242 Hz. The impedance minimum near 50 Hz is not a resonant frequency even though the velocity and pressure profiles look very similar to the quarter wavelength resonance calculated for the transmission line enclosure. One last thought; thinking about a bass reflex system, is the bass produced by the port at the impedance minimum where driver motion essentially stops a resonance? Hopefully looking back at the preceding discussion and after thinking a little the answer should be obvious. If not, the answer is included in the conclusions section.

Figure 19 : Pressure and Velocity Profiles at Selected Frequencies of the Electrical Impedance Curve for a Straight Transmission Line (Stuffing Density = 0.0 lb/ft^3)

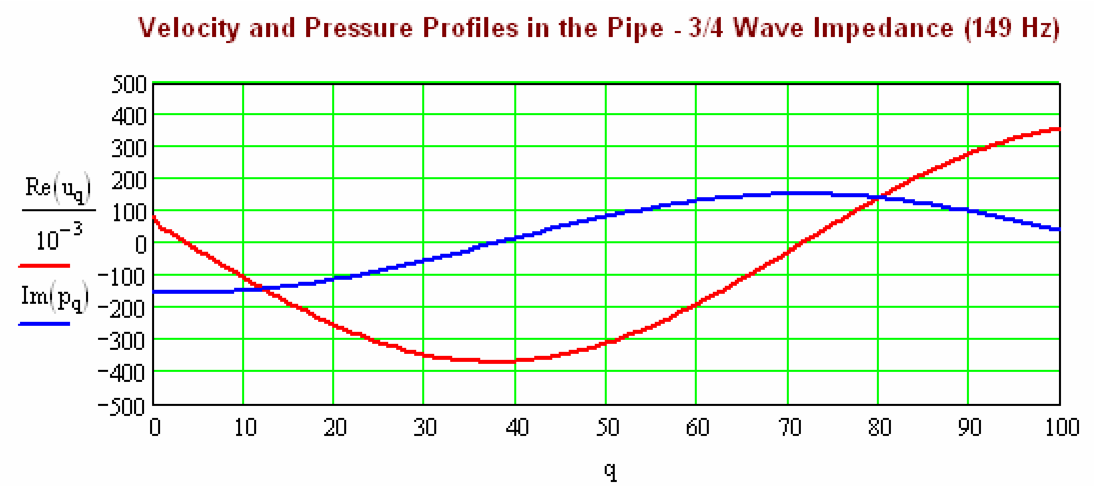






(continued)

Figure 19 (continued)



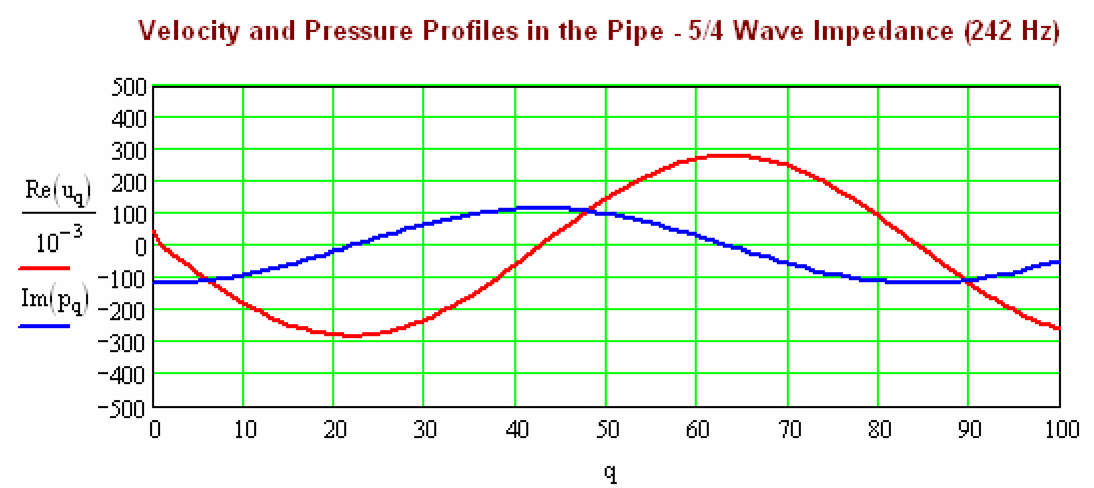
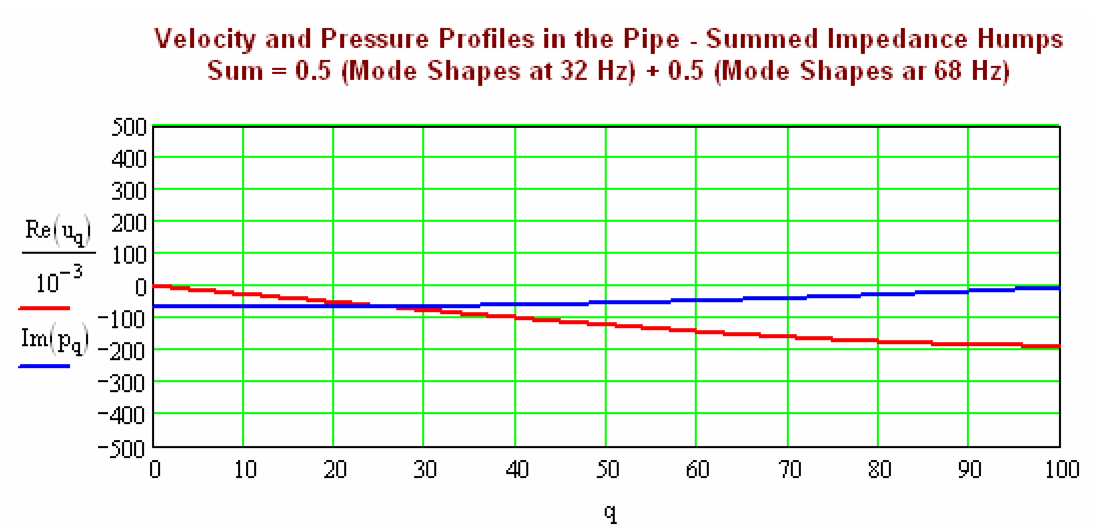


Figure 20 : Summed Mode Shapes for a Straight Transmission Line



Adding the Fiber Stuffing :

The stuffing density used in many older rule of thumb transmission line designs is 0.5 lb/ft^3 for the entire length of the line. Adding stuffing to the model and rerunning the MathCad simulation produces the plots shown in Figure 21. The first thing that jumps out is how much smoother the system SPL curve is in the top plot. This plot is representative of classic transmission line performance which produces a ripple in the pass band when the driver is mounted at the closed end. The middle plot does not contain any of the sharp peaks and nulls in the driver and open end SPL responses. And finally, the impedance curve is now a single humped curve which many people associate with a correctly stuffed transmission line.

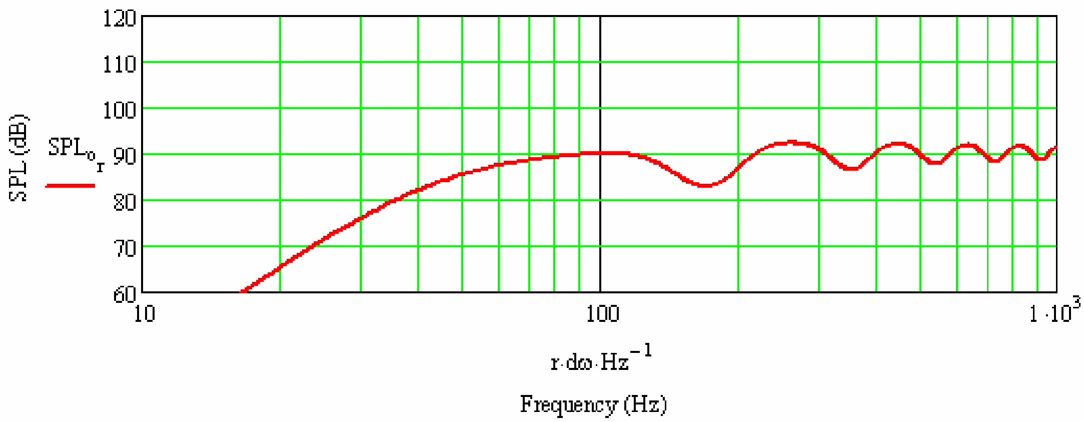
To see what is taking place inside the transmission line enclosure, the pressure and velocity profiles are plotted in Figure 22. Comparing the curves in Figure 22 with those presented in Figure 19 the reader can observe the impact the stuffing has on damping the standing waves. Believe it or not, the input to the driver is identical in both sets of plots.

Again, starting with the higher frequency model shapes at 149 and 242 Hz, the mode shapes look very similar to the last two plots in Figure 19. However, the fiber damping has significantly reduced the amplitude of the response eliminating the peaks at these frequencies seen in the middle plot of Figure 18.

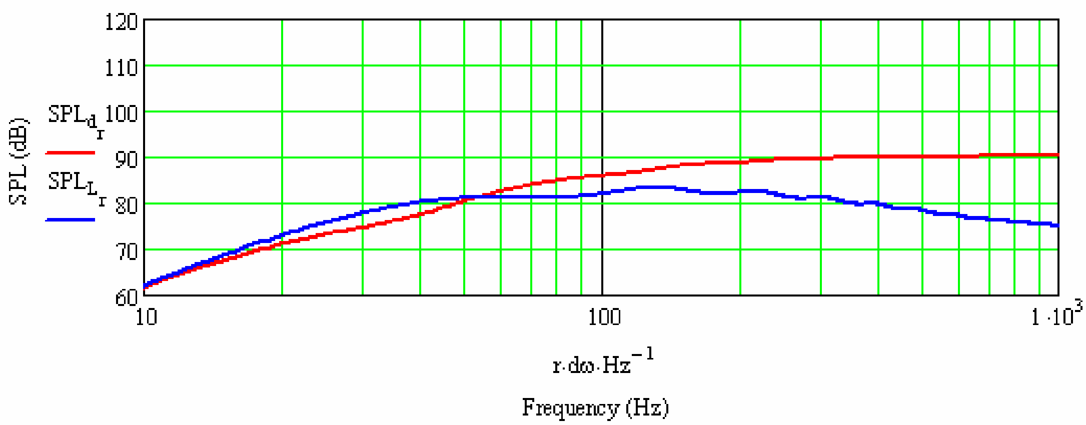
The three plots that correspond to the double humps and the minimum between them have also changed significantly. The first plot at 32 Hz exhibits an almost constant velocity along the length which leads to a zero pressure condition. The air in the transmission line is moving as a solid slug at 32 Hz essentially adding the equivalent air mass to the driver's moving mass. Since this resonance is so well damped, the first hump in the impedance curve is eliminated. The velocity and pressure curves at 50 Hz and 68 Hz are similar to the previous curves just attenuated slightly. Again, the velocity and pressure curves at 50 Hz can be expressed as a linear combination of the curves plotted at the 32 and 68 Hz resonances.

Figure 21 : Straight Transmission Line System Simulation Results
(Stuffing Density = 0.5 lb/ft^3)

Far Field Transmission Line System Sound Pressure Level Response



Woofer (red curve) and Open End (blue curve) Far Field Sound Pressure Level Responses



Transmission Line System Electrical Impedance

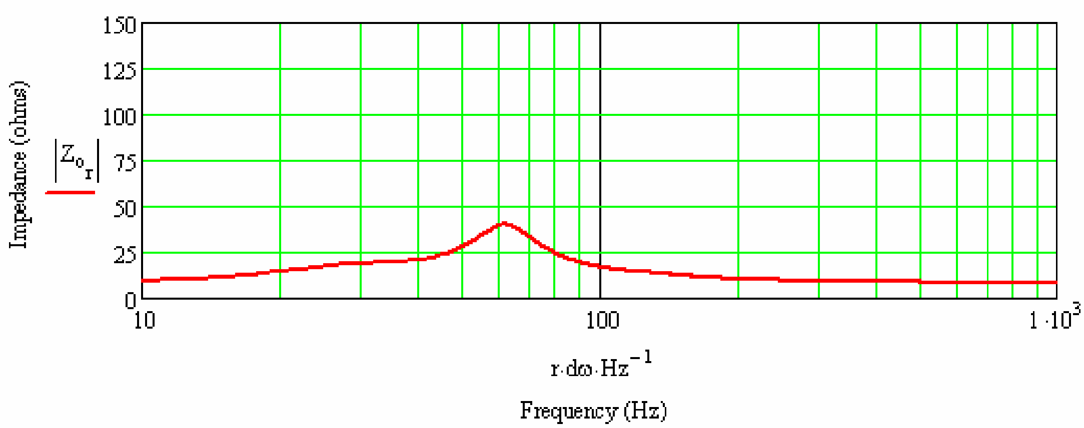
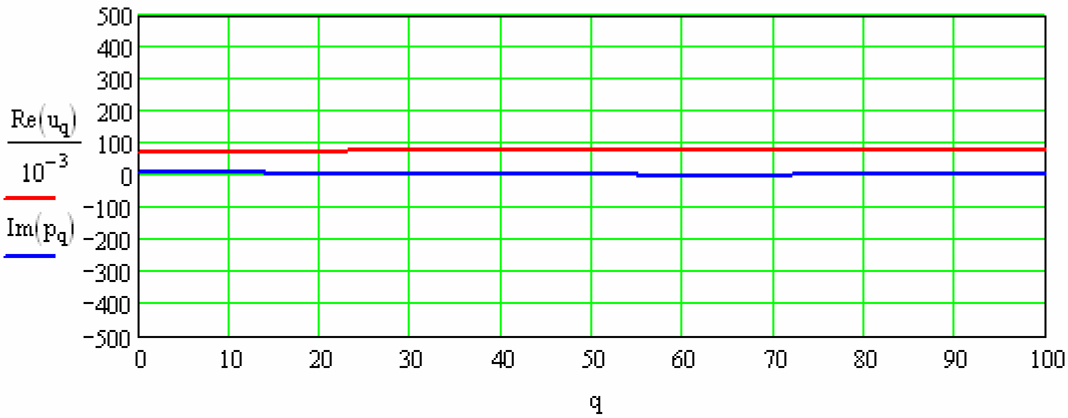
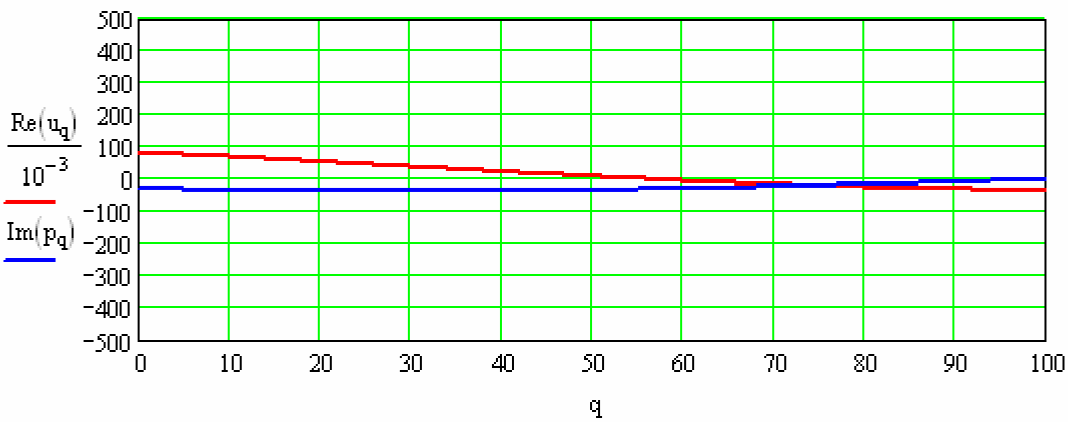


Figure 22 : Pressure and Velocity Profiles at Selected Frequencies of the Electrical Impedance Curve for a Straight Transmission Line (Stuffing Density = 0.5 lb/ft^3)

Velocity and Pressure Profiles in the Pipe - First Impedance Peak (32 Hz)



Velocity and Pressure Profiles in the Pipe - Impedance Null (50 Hz)



Velocity and Pressure Profiles in the Pipe - Second Impedance Peak (68 Hz)

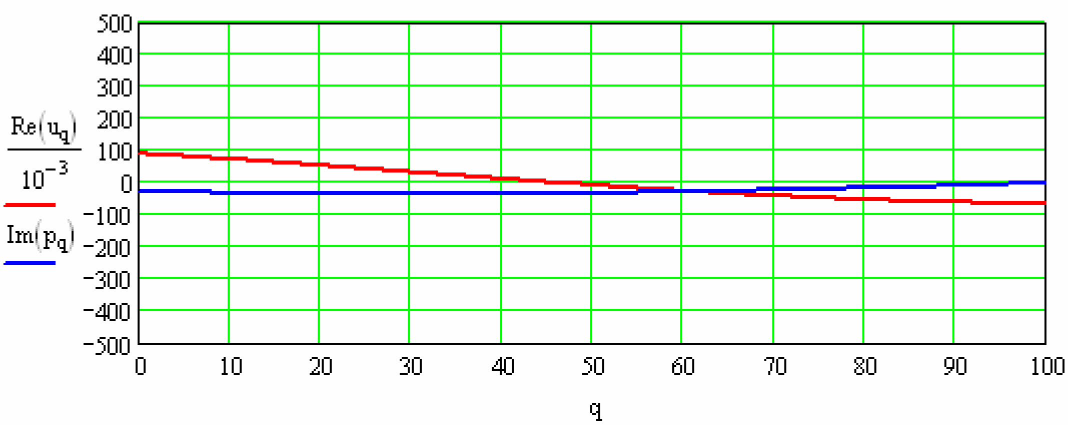


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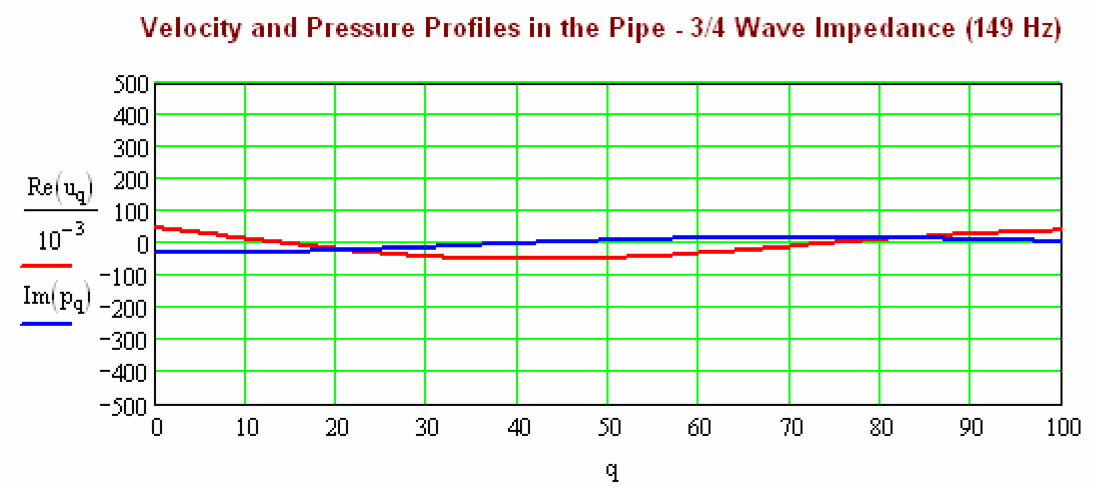
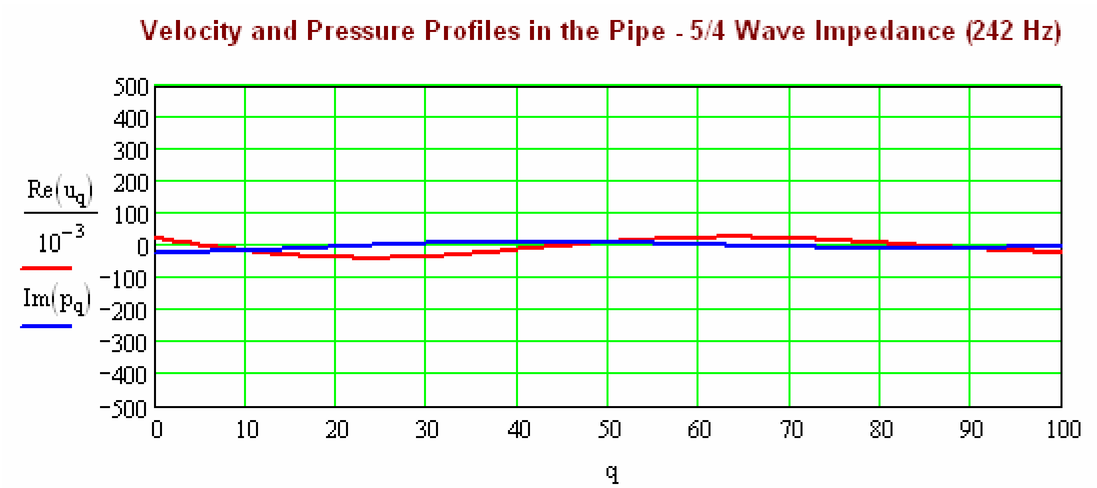
Figure 22 (continued)





Tapered and Expanding Transmission Line Systems :

Probably one of the most common mistakes people make when designing a transmission line speaker system involves determining the appropriate line length for a tapered or expanding (TQWT) enclosure geometry. The length required for the same tuning frequency is different for tapered, straight, or expanding enclosure geometries. To demonstrate this concept, two additional enclosures will be sized, analyzed, and results presented in a similar manner to what was described in the previous sections. The three geometries are shown below where the straight case has already been shown. Each enclosure's length was determined for the simple open end boundary condition so the final tuning will be slightly lower after the open end's acoustic impedance is introduced.

Variable	Tapered	Straight	Expanding
Length (in)	48.5	67.3	87.5
S_0	5 x S_d	2 x S_d	1 x S_d
S_L	1 x S_d	2 x S_d	5 x S_d

Comparing Figures 15, 23, and 26 it can be seen that the fundamental tuning frequency is approximately 50 Hz, modified by the more complex open end acoustic impedance boundary condition, but that the harmonics occur at different frequency intervals. For the shorter tapered geometry the next resonance is higher in frequency while for the longer expanding geometry the next resonance is lower in frequency when compared to the results for the straight transmission line geometry. This is a key concept often missed.

Comparing the bottom plot in Figures 15, 23, and 27 it is also clear that the number of resonances that will impact the combined system response is less for the tapered transmission line enclosure. Figure 23 shows only three well damped peaks below 500 Hz. These observations carry over into the plots shown in Figures 16, 24, and 28. Adding fiber stuffing to the tapered and expanding transmission line speaker systems produces the responses plotted in Figures 26 and 30. Reviewing the final transmission line SPL responses shown in the top curves of Figures 21, 26, and 30 a strong argument can be made for the advantages of tapered transmission line designs.

One final point worth noting is seen in the mode shape plots of Figures 19, 25, and 29. For the straight transmission line the 3/4 and 5/4 wavelength mode shapes are typical of odd multiples of the quarter sine and cosine waves. However for the tapered and expanding geometries, these mode shapes start to become a little distorted and are not as close to perfect quarter wavelength shapes. For the tapered transmission line, the first two resonances show a distorted shape with the velocity increasing significantly at the open end. For the expanding transmission line, the velocity distortion occurs at the opposite end. Please note that by distortion I mean deviation from the perfect sine or cosine wave and not audible distortion of the signal being reproduced.

Summing up what can be observed for these two new geometries. Sizing the length of a tapered and expanding transmission line is not as simple as the straight transmission line. All of the behaviors noted for the straight transmission line system with respect to combining driver and enclosure systems also apply to these geometries. And finally, after reviewing Figures 21, 26, and 30 I can see real advantages for tapered transmission line enclosure geometries if a classic transmission line is to be designed and constructed.

Figure 23 : Tapered Transmission Line Enclosure Acoustic Impedance
Impedance Boundary Condition at x = L

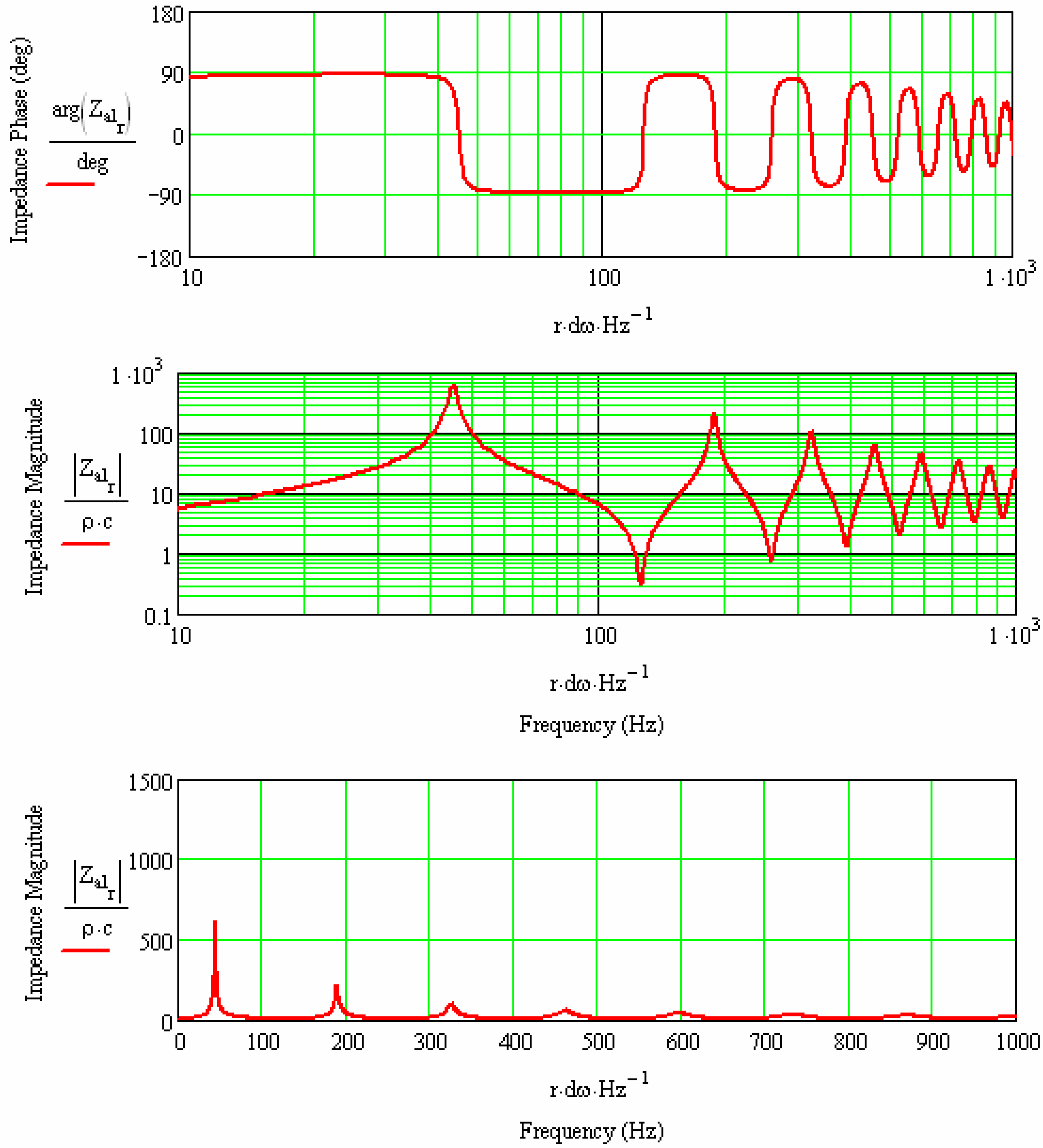
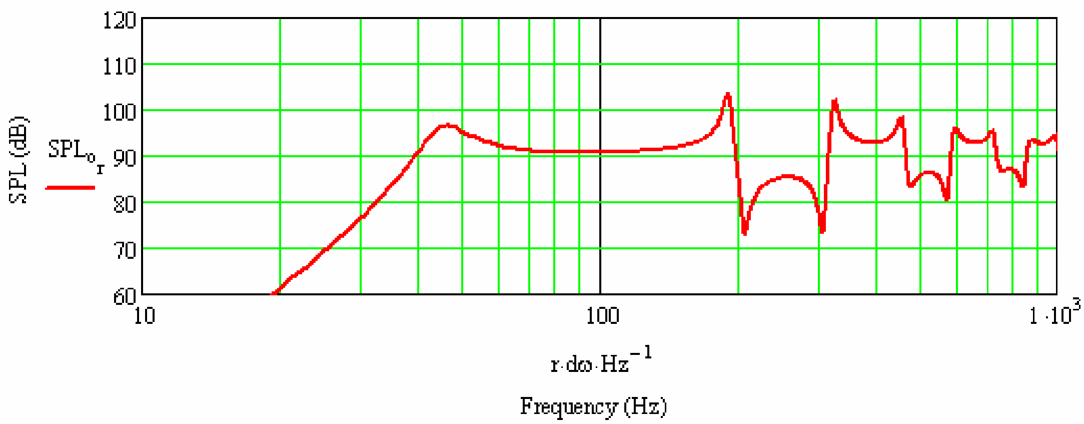
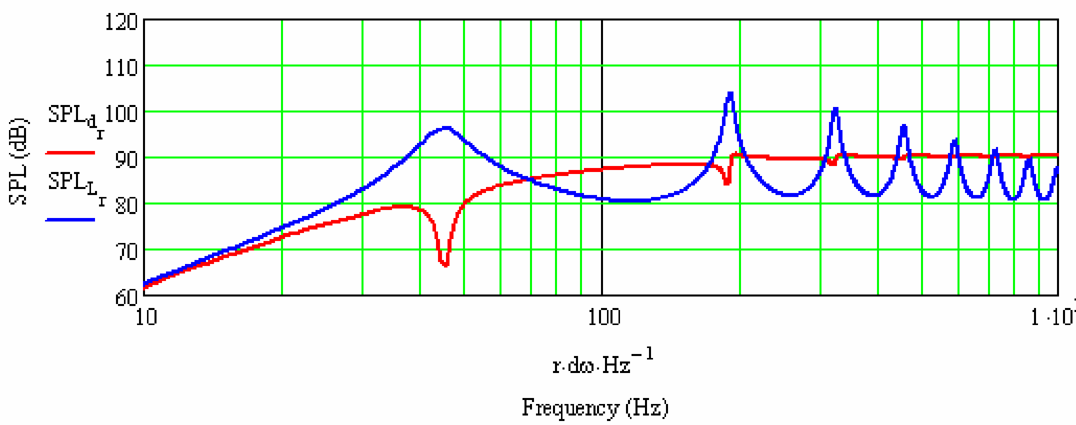


Figure 24 : Tapered Transmission Line System Simulation Results
(Stuffing Density = 0.0 lb/ft^3)

Far Field Transmission Line System Sound Pressure Level Response



Woofer (red curve) and Open End (blue curve) Far Field Sound Pressure Level Responses



Transmission Line System Electrical Impedance

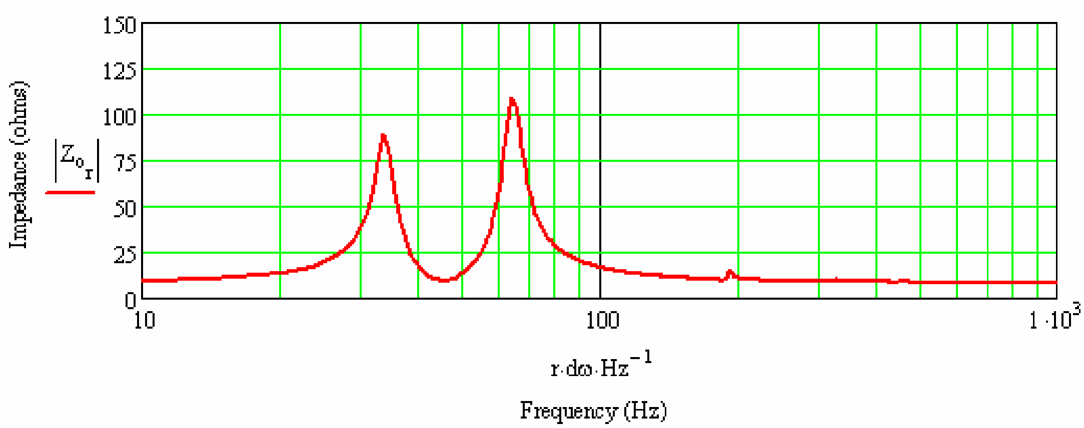
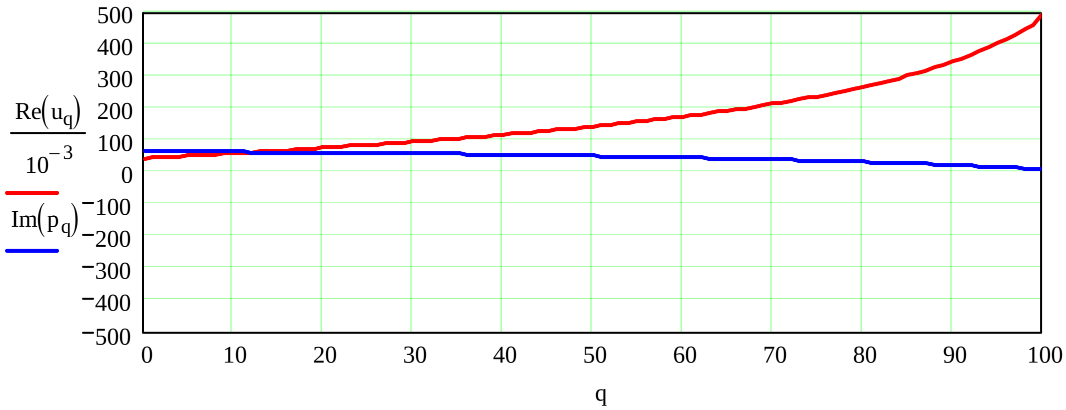
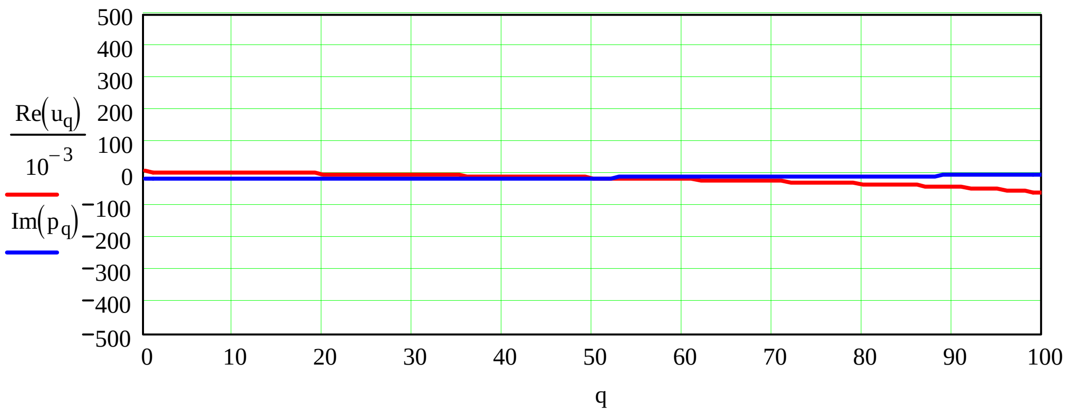


Figure 25 : Pressure and Velocity Profiles at Selected Frequencies of the Electrical Impedance Curve for a Tapered Transmission Line (Stuffing Density = 0.0 lb/ft^3)

Velocity and Pressure Profiles in the Pipe - First Impedance Peak (34 Hz)



Velocity and Pressure Profiles in the Pipe - Impedance Null (46 Hz)



Velocity and Pressure Profiles in the Pipe - Second Impedance Peak (65 Hz)

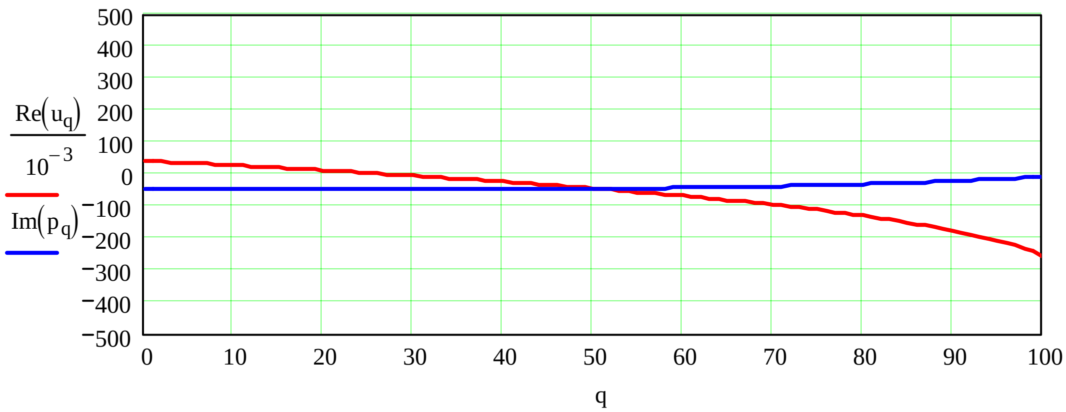


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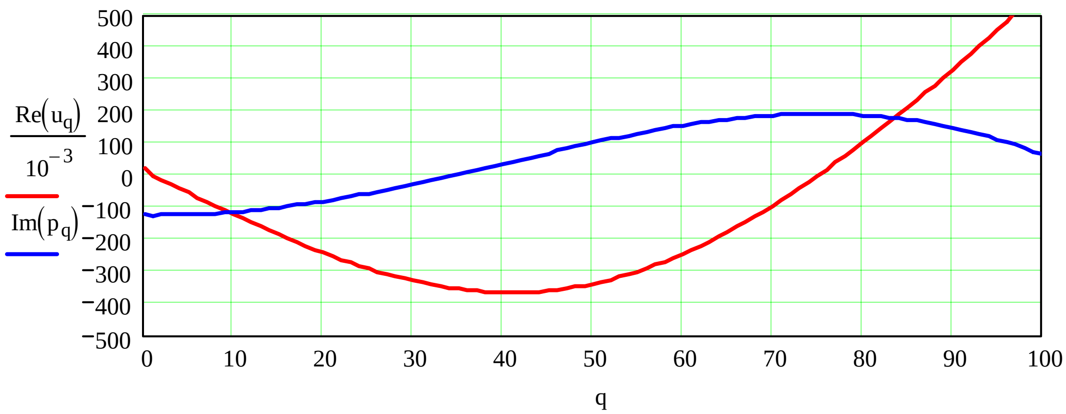
Figure 25 : (continued)

Velocity and Pressure Profiles in the Pipe - 3/4 Wave Impedance (194 Hz)



Velocity and Pressure Profiles in the Pipe - 5/4 Wave Impedance (328 Hz)

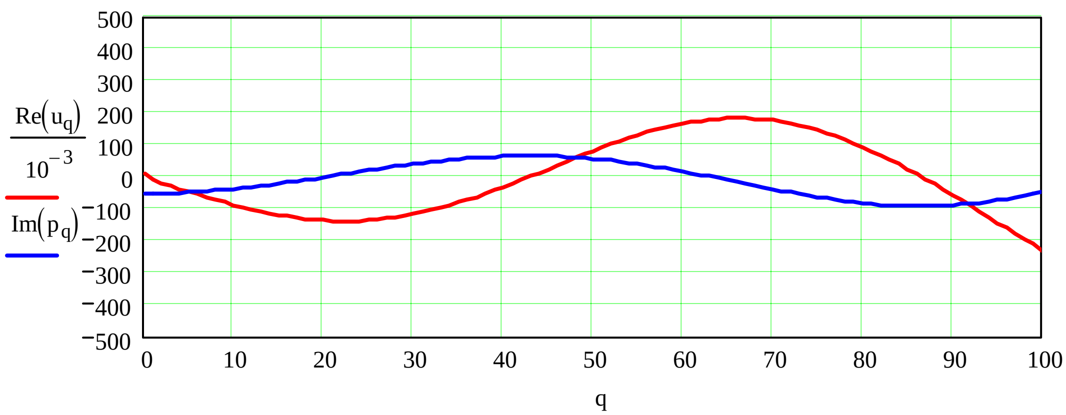
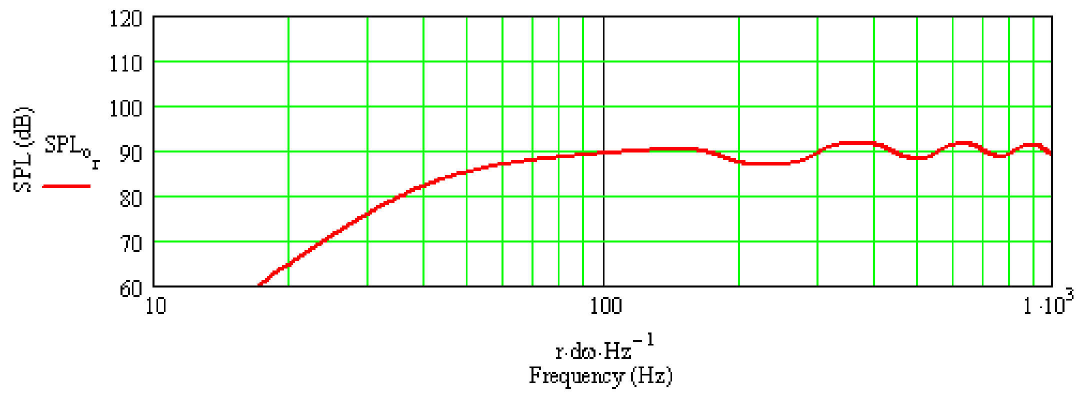
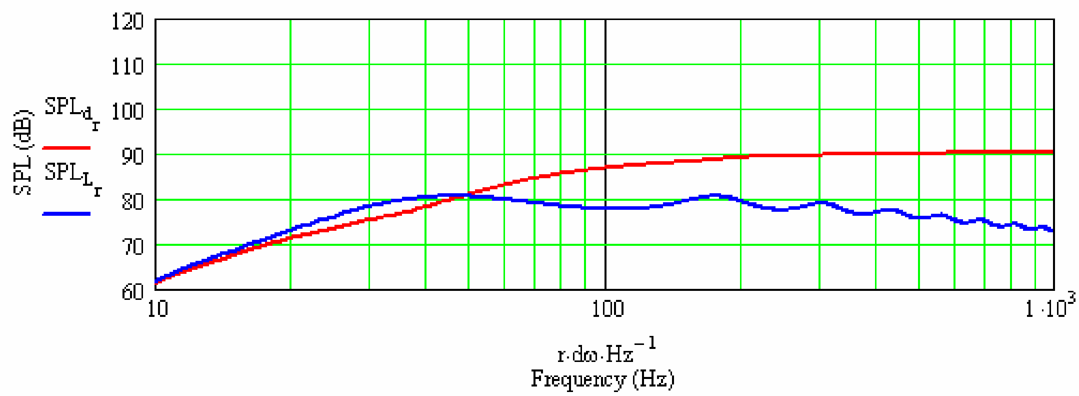


Figure 26 : Tapered Transmission Line System Simulation Results
(Stuffing Density = 0.5 lb/ft^3)

Far Field Transmission Line System Sound Pressure Level Response



Woofer (red curve) and Open End (blue curve) Far Field Sound Pressure Level Responses



Transmission Line System Electrical Impedance

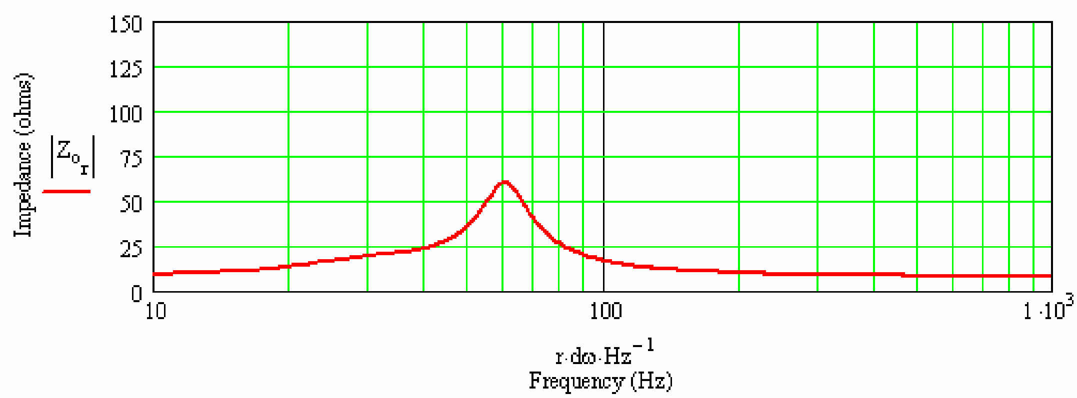


Figure 27 : Expanding Transmission Line Acoustic Impedance
Impedance Boundary Condition at x = L

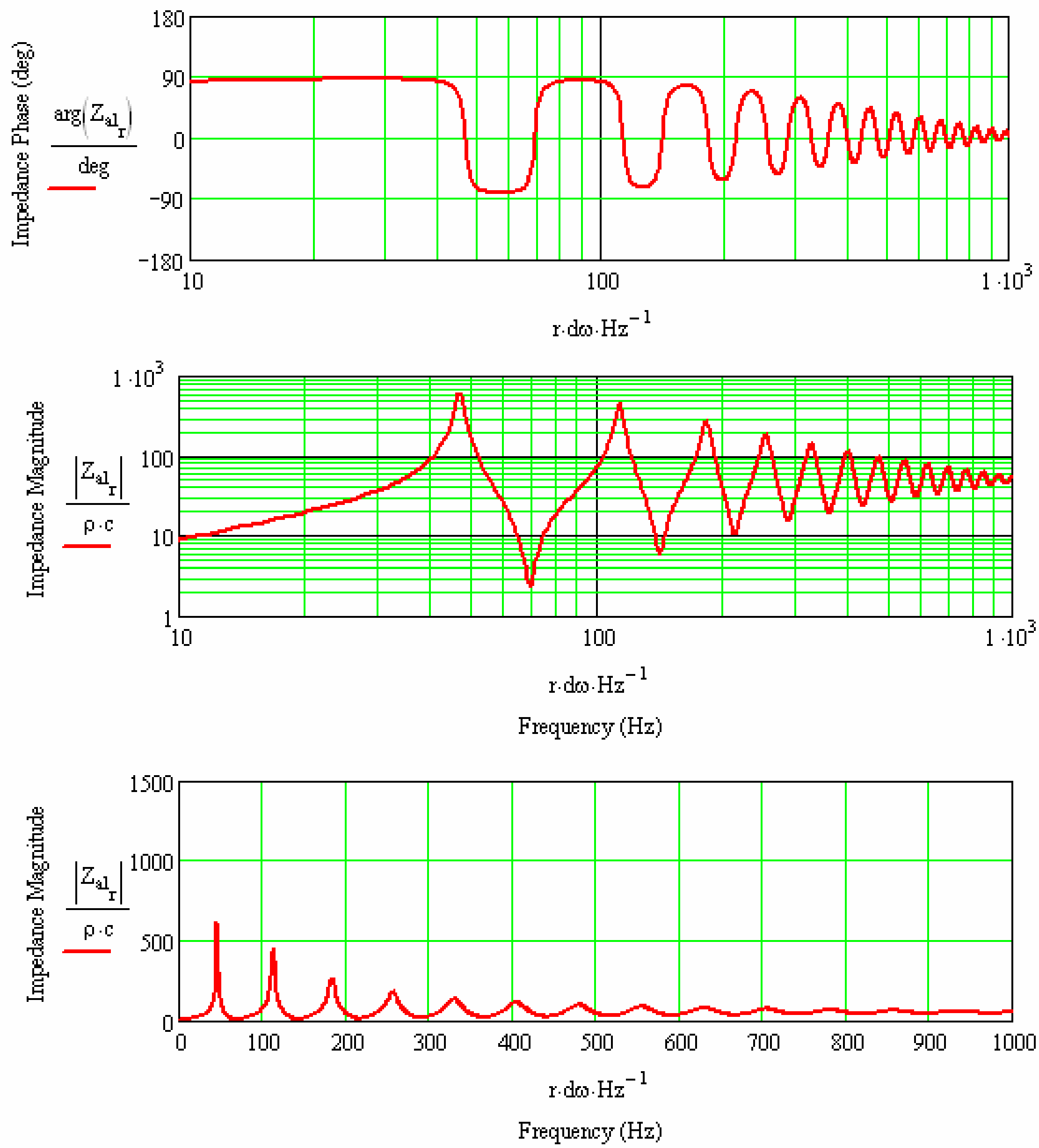
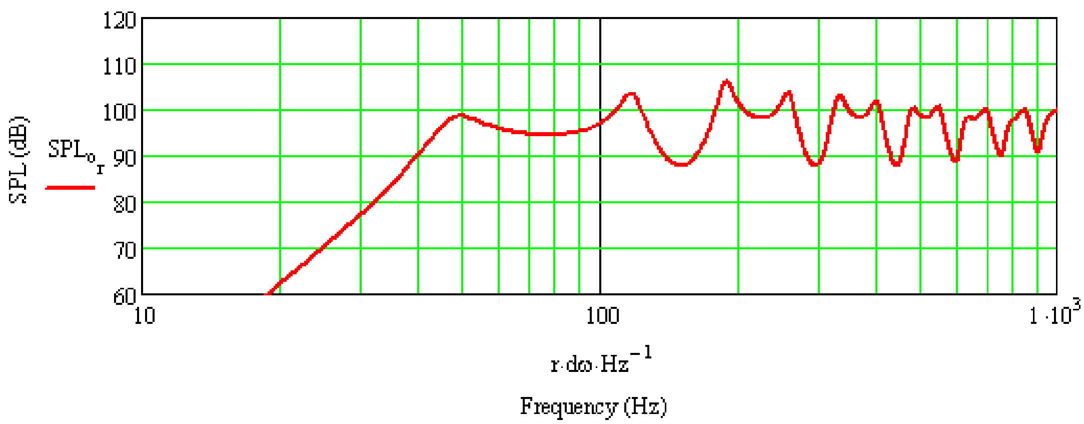
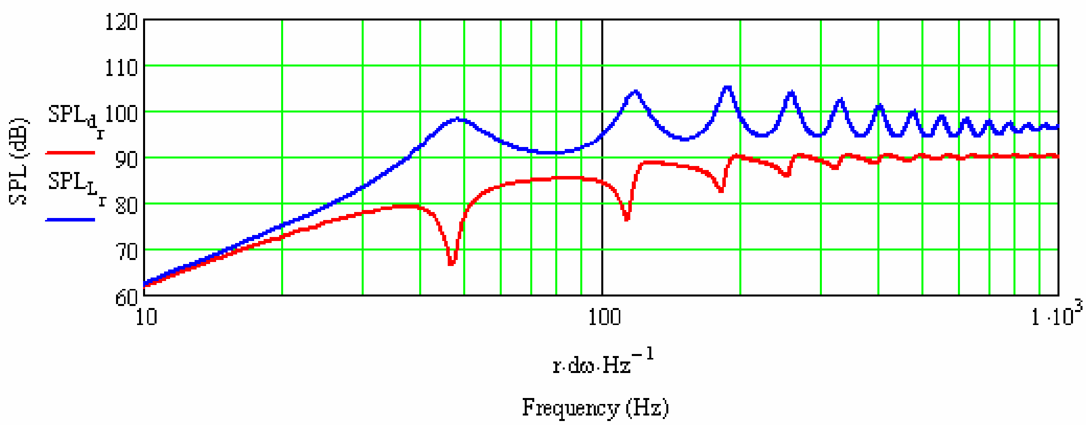


Figure 28 : Expanding Transmission Line System Simulation Results
(Stuffing Density = 0.0 lb/ft^3)

Far Field Transmission Line System Sound Pressure Level Response



Woofer (red curve) and Open End (blue curve) Far Field Sound Pressure Level Responses



Transmission Line System Electrical Impedance

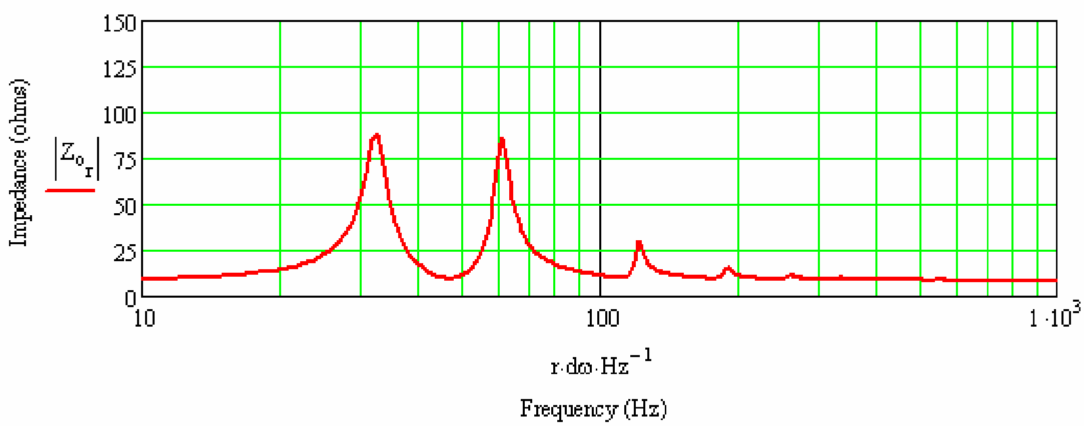
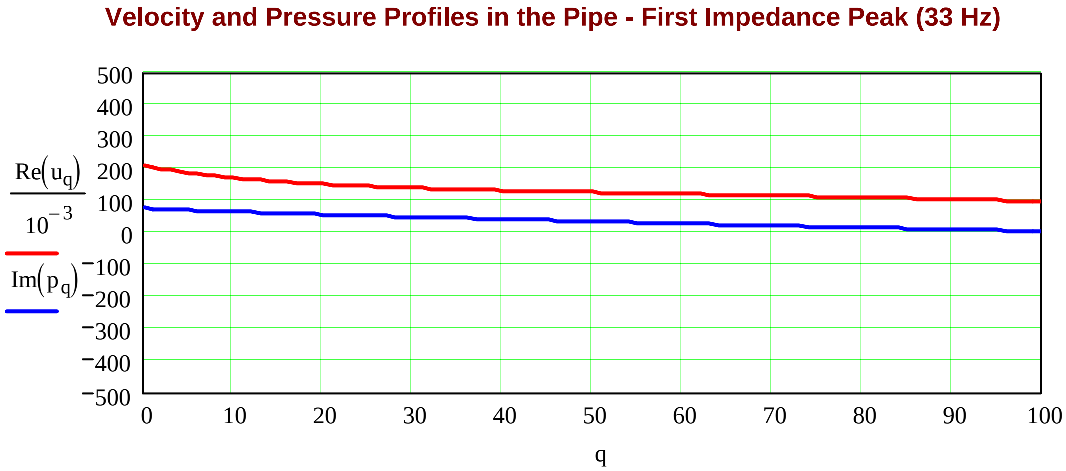
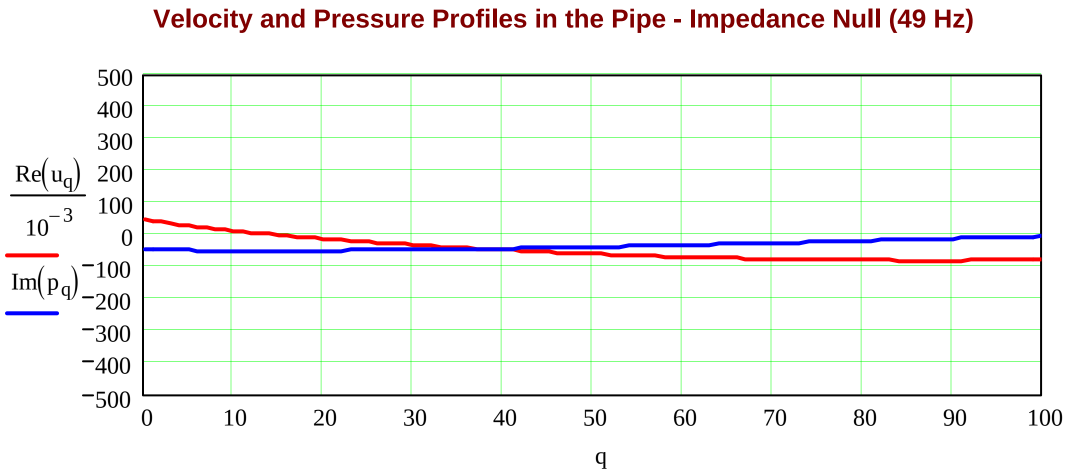
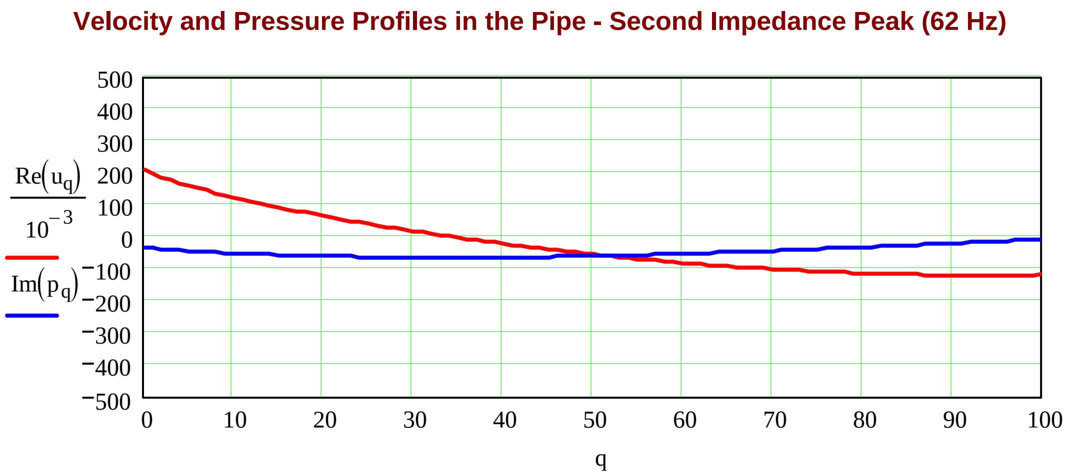


Figure 29 : Pressure and Velocity Profiles at Selected Frequencies of the Electrical Impedance Curve for an Expanding Transmission Line (Stuffing Density = 0.0 lb/ft^3)

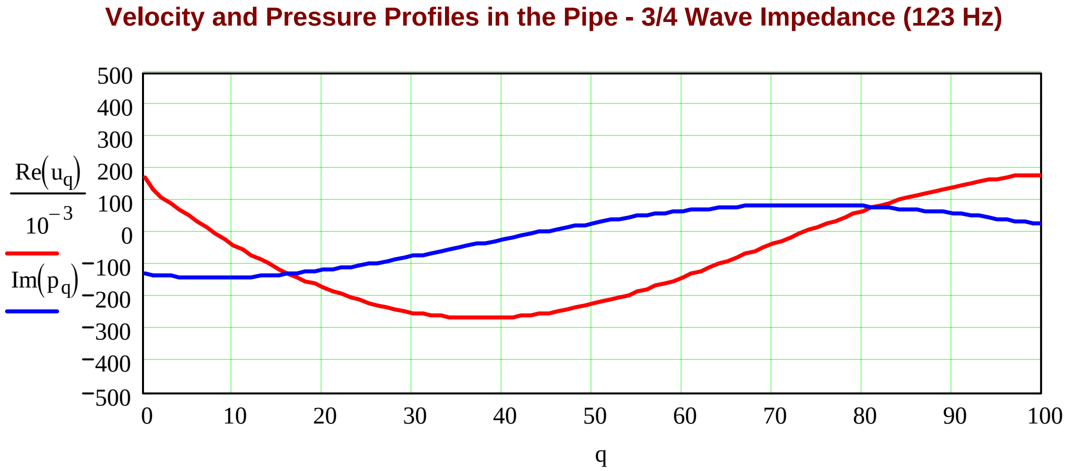






(continued)

Figure 28 : (continued)



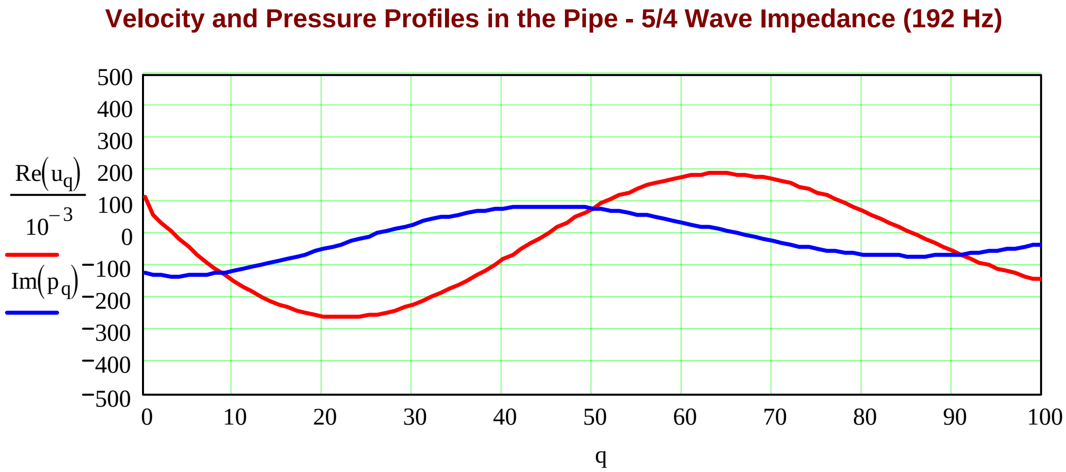
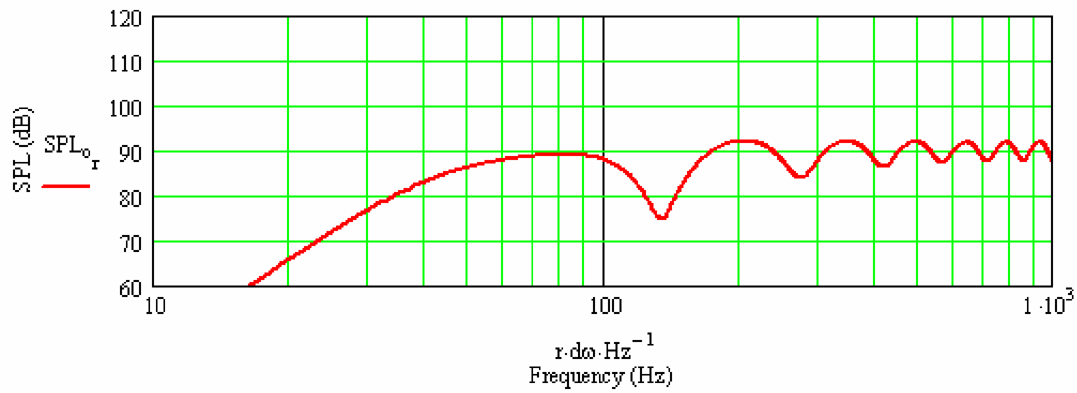
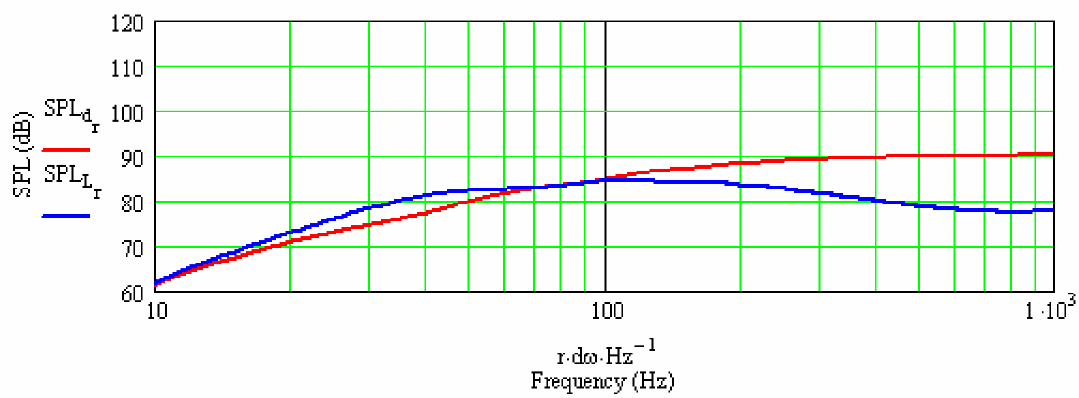


Figure 30 : Expanding Transmission Line System Simulation Results
(Stuffing Density = 0.5 lb/ft^3)

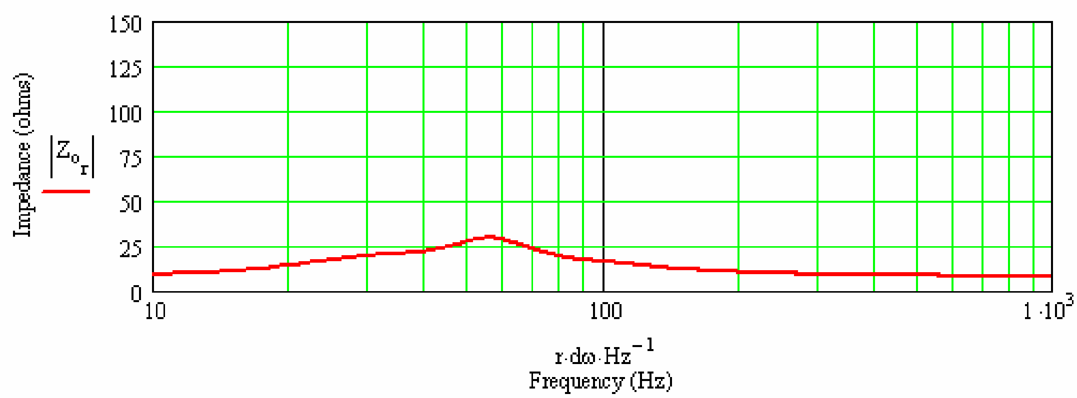
Far Field Transmission Line System Sound Pressure Level Response



Woofer (red curve) and Open End (blue curve) Far Field Sound Pressure Level Responses



Transmission Line System Electrical Impedance



Conclusions :

Probably the most important point to remember is that the lengths of tapered, straight, and expanding transmission line enclosure are not the same for a give tuning frequency. The tapered transmission line will be the shortest while the expanding line will be the longest. If somebody is using the well known equation :

$$L = c / (4 f_n)$$

where

L = length in meters
c = speed of sound, typically assume to be 343 m/sec
f_n = tuning frequency in Hz

to determine the length of a tapered or expanding geometry, their design is in trouble right out of the gate. Consulting Table 1 of my *Classic Transmission Line Enclosure Alignment Tables*[7] article will provide a better estimate for the required length of tapered or expanding transmission line geometries.

The double humped impedance curve is an inherent property of the transmission line speaker system. Combining a driver with a transmission line enclosure, when both have approximately the same fundamental resonant frequencies, results in two new system resonances. There will be one resonance below and one above the original independent resonant frequencies. This is the source of the double humped electrical impedance curve. To answer the question posed earlier, the same phenomenon occurs in a bass reflex speaker system.

At the minimum between the two electrical impedance peaks, the enclosure will produce most of the sound while the driver's motion is a minimum. By definition this is not a system resonance. It is the linear combination of the system resonances that exist and produce the double humped impedance curve. The quarter wavelength standing wave at the enclosure tuning frequency still exists but it is not defined as a system resonant condition. This concept is probably not commonly recognized by the DIY speaker builder and will most likely raise a few eyebrows.

Hopefully, the reader will come away from this article understanding a couple of significant properties of transmission line speaker systems a little better. I have tried to include a little math, some analogies as explanation, and a lot of pictures so that the widest possible audience can find the information in this article useful. As always, I am open to comments and suggestions that would improve the article. Any questions would also be welcomed and help me determine where my attempts at explaining how a transmission line speaker system works need additional details and discussion.

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7) Classic Transmission Line Enclosure Alignment Tables by Martin J. King dated 10/16/03, www.quarter-wave.com